Last time. Proved that an adjunction \( (\delta \xleftarrow{\varepsilon} \Gamma, \eta : \Delta \Rightarrow \Phi, \zeta : H \Rightarrow G) \) gives rise to a monad \( \Pi = (T = GF, \eta, \delta = G \varepsilon F) \).

- Defined the Kleisli category \( K_T \) of a monad \( T \) on a category \( C \):
  - The objects of \( K_T \) are the objects of \( C \), a morphism \( a \to b \) in \( C \) gives rise to a monad \( \Pi \) on a category \( f \):
    - The objects of \( f \) are the objects of \( C \), a morphism \( a \to b \) in \( C \) is a morphism \( a \to T b \) in \( B \), and the composition \( \cdot \) is defined by
      \[
      (\cdot)_{a \to b} \cdot (\cdot)_{b \to c} = \cdot_{a \to c}
      \]
      where on the right the composition is in \( C \).

Example. Let \( C = \text{Set} \), \( M \) a monoid and \( \Pi = (T, \eta, \delta) \) the corresponding writer monad: \( T (x, y) = (x \times M, y \times M) \) for all \( x, y \) in \( \text{Set} \).

The multiplication \( \mu : T^2 \Rightarrow T \) is defined by setting
\[
\mu(a, b) (x, y) = (x \cdot a \cdot b)
\]

A Kleisli morphism for this monad, that is, a morphism in the Kleisli category \( \text{Set}_T \), is a function \( k : X \to Y \times M \). For example if \( M \) is a finite list monoid, a morphism \( k : X \rightsquigarrow Y \) would model a process that computes a certain value and, at the same time, writes a string of symbols in a log file.

The Kleisli composition of \( k : X \to Y \times M \) and \( h : Y \to Z \) is
\[
X \xrightarrow{k} Y \times M \xrightarrow{h \times \text{id}_M} Z \times M \xrightarrow{\mu_{Z \times M}} Z \times M
\]

\( x \mapsto (h_{k1(x)}, h_{k2(x)}) \mapsto (h_1(k_1(x)), h_2(k_1(x)), k_2(x)) \mapsto (h_1 \circ k_1)(x), h_2(k_1(x)) \cdot k_2(x) ) \)

Again, if \( M \) is a finite list monoid and \( k, h \) are processes that compute values and write to the same log file then their Kleisli composite would concatenate the strings.

The Kleisli adjunction \( C \xrightarrow{\eta} C_T \)

Let \( \Pi = (T, \eta, \delta) \) be a monad on a category \( C \). We define a functor
\[
F_T : C \to C_T \text{ by } F_T(a) = a \text{ for all objects } a \text{ of } C \text{ and } F_T(a \cdot f \cdot b) = a \cdot T f \cdot b \cdot \Phi T b
\]
for all morphisms $a \to b$ in $\mathcal{C}$. Since $F_{\pi}(Ha) = a \xrightarrow{Ta} a$ for all $a \to b$, $F_{\pi}$ preserves identities. We need to check that $F_{\pi}$ preserves composition. Given $c \to b \to a \in \mathcal{C}$,

$$(F_{\pi} g) \circ \kappa(f\pi(f)) = a \to b \xrightarrow{g \circ f} c \xrightarrow{g \circ f \circ Tg} Tc \xrightarrow{id_{Tc}} Tc = F_{\pi}(g \circ f)$$

Hence $F_{\pi} : \mathcal{C} \to \mathcal{E}_{\pi}$ is a functor.

Next we define a functor $G_{\pi} : \mathcal{E}_{\pi} \to \mathcal{C}$. We'll see that $G_{\pi}$ is right adjoint to $F_{\pi}$ and that the monad defined by the adjunction $F_{\pi} \dashv G_{\pi}$ is the monad $\mathcal{T}$ we're starting with.

On objects we define $G_{\pi}$ by $G_{\pi}(a) = Ta$ for all $a \in \mathcal{C}$. Given $a \xrightarrow{f} b$ in $\mathcal{C}$ (i.e. $a \to b$ in $\mathcal{C}$), we define $G_{\pi}(f) = Tb \xleftarrow{Mb} TTb \xrightarrow{Tf} Ta$; that is, $G_{\pi}(a \xrightarrow{f} b) = Ta \xrightarrow{Mb \circ Tf} Tb$.

$G_{\pi}(a \xrightarrow{f} b) = Mb \circ Tf$ and $G_{\pi} \circ f$ is by the right unit identity for the monad:

$$G_{\pi}(f) = \xrightarrow{Mb \circ Tf}$$

It remains to check that $G_{\pi}$ preserves composition. Consider $c \to b \to a \in \mathcal{C}$.

$G_{\pi}(g \circ f) = G_{\pi}(g) \circ G_{\pi}(f) = G_{\pi}(g \circ f) \circ G_{\pi}(g) = G_{\pi}(g \circ f \circ Tg) \circ Tg$.

The diagram commutes:

Hence $G_{\pi}(g \circ f) = G_{\pi}(g) \circ G_{\pi}(f)$. $G_{\pi}$ is a functor.

**Proposition 38.1** Let $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ be a monad on a category $\mathcal{C}$, $\mathcal{C}_{\pi}$ the Kleisli category and $F_{\pi} : \mathcal{C} \to \mathcal{C}_{\pi}$, $G_{\pi} : \mathcal{C}_{\pi} \to \mathcal{C}$ the functors defined above.
\[
i) \quad G_T \circ F_T = \text{id} \\
ii) \quad F_T \dashv G_T \\
iii) \quad \text{The unit of the adjunction } F_T \dashv G_T \text{ is } \eta, \text{ the unit of the monad } \mathbb{T} \\
v) \quad G_T \circ F_T = \mu, \text{ and consequently the adjunction } F_T \dashv G_T \text{ recovers } \mathbb{T}.
\]

**Proof**

(i) On objects, \( G_T(F_T(a)) = G_T(a) = Ta \) for all \( a \in \mathcal{C} \).

\[
G_T(F_T(a \xrightarrow{f} b)) = G_T(a \xrightarrow{f} Tb) = Ta \xrightarrow{Tf} Tb \xrightarrow{T\eta_b} TTTb \xrightarrow{\mu_b} Tb = Ta \xrightarrow{Tf} Tb.
\]

(ii) For any objects \( a, b \in \mathcal{C} \), a Kleisli morphism \( a \xrightarrow{f} b \) is a morphism \( a \xrightarrow{\eta} Tb \) in \( \mathcal{C} \).

Hence \( \text{Hom}_{\mathcal{E}}(F_T(a), b) = \text{Hom}_{\mathcal{P}}(a, b) = \text{Hom}_{\mathcal{C}}(a, Tb) = \text{Hom}_{\mathcal{C}}(a, G_T(b)) \) since \( a \in \mathcal{C} \), \( F_T(a) = a \) and \( b \in \mathcal{E} \), \( G_T(b) = Tb \in \mathcal{E} \).

We need to check that \( \Theta_{a,b} = \eta : \text{Hom}_{\mathcal{E}}(F_T(a), b) \to \text{Hom}_{\mathcal{C}}(a, G_T(b)) \) is natural in \( a \) and \( b \).

Given \( g : a' \to a \), we check that

\[
\begin{align*}
\text{Hom}_{\mathcal{E}}(F_T(a), b) & \xrightarrow{g^*} \text{Hom}_{\mathcal{C}}(a, G_T(b)) \\
\text{Hom}_{\mathcal{E}}(F_T(a'), b) & \xrightarrow{g'^*} \text{Hom}_{\mathcal{C}}(a', G_T(b))
\end{align*}
\]

commutes.

Since \( \text{Hom}_{\mathcal{E}}(F_T(a), b) = \text{Hom}_{\mathcal{C}}(a, Tb) \), \( F_T(g)^* k = k \circ \eta_b \circ F_T(g) = \mu_b \circ Tk \circ (\eta_a \circ g) \).

\( F_T(g)^* k = \mu_b \circ Tk \circ \eta_a \circ g = k \circ g = g^* (k) \).

(iii) For any \( h : b \to b' = b \to Tb \), we check that

\[
\begin{align*}
\text{Hom}_{\mathcal{E}}(F_T(a), b) & \xrightarrow{h^*} \text{Hom}_{\mathcal{C}}(a, Tb) \\
\text{Hom}_{\mathcal{E}}(F_T(a), b') & \xrightarrow{g'^*} \text{Hom}_{\mathcal{C}}(a, Tb')
\end{align*}
\]

commutes.

For any \( k : a \to b \in \text{Hom}_{\mathcal{E}}(F_T(a), b) = \text{Hom}_{\mathcal{C}}(a, Tb) = \text{Hom}_{\mathcal{C}}(a, G_T(b)) \)
\[ h^* k = h^* \pi k = \mu_b \circ T h \circ k, \quad \text{while} \quad G_T (h)^* k = G_T (h) \circ k = (\mu_b \circ T h) \circ k \]
as well.

(iii) \( \text{id}_{T_!: a} : F_T (a) \to F_T (a) \) in the Kleisli category \( E_T \), \( \eta : a \to T a \) in \( \text{Hom}_{E_T} (a, G_T (a)) \). Hence the unit of the adjunction \( E \Rightarrow G_T \circ F_T \) is precisely \( \eta \).

(iv) \( \text{id}_{G_T (a)} : G_T (a) \to G_T (a) \) corresponds to \( \varepsilon_a : F_T \circ G_T (a) \to a \). Hence \( \varepsilon_a : T a \to a \) is \( \text{id}_{T a} \).

(v) \( (G_T \circ F_T) a = G_T (E F (a)) = G_T (T a \to a) = G_T (T a \downarrow T a) = \mu_a \circ T (\text{id}_{T a}) = \mu a \)

Recall \( G_T (a \downarrow b) = G_T (a \downarrow T b) = \mu_b \circ T f \).

There is more to say about monads and about category theory more generally, but we’re out of time.