Monads

"A monad is just a monoid in the category of endo-functors, what's the problem?"

James Iry paraphrasing MacLane and attributing it to Philip Wadler.

First we revisit the definition of a monoid. Recall that a monoid is a triple $(M, \mu, e)$ where $M$ is a set, $\mu : M \times M \to M$, $\mu(a, b) = a \cdot b$ in a function (multiplication) and $e \in M$ is an element (unit) so that

i) $\mu$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in M$.

ii) $e$ is a unit for $\mu$: $e \cdot a = a = a \cdot e$ for all $a \in M$.

We can rewrite (i) as $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ which says that the diagram

\[
\begin{array}{c}
M \times M \\
\downarrow \\
M
\end{array}
\xrightarrow{\mu} 
\begin{array}{c}
M \\
\downarrow \\
M
\end{array}
\]

commutes.

We can think of $e$ as a function $e : 1 \to M$ where $1 = * \in$ a one element set. Then (ii) says $\mu \circ (e \times \text{id}_M) = \text{id}_M = \mu \circ (\text{id}_M \times e)$, i.e. the diagrams

\[
\begin{array}{c}
1 \times M \\
\downarrow \\
M
\end{array}
\xrightarrow{e \times \text{id}_M} 
\begin{array}{c}
M \times M \\
\downarrow \\
M
\end{array}
\]

and

\[
\begin{array}{c}
M \times M \\
\downarrow \\
M
\end{array}
\xleftarrow{\text{id}_M \times e} 
\begin{array}{c}
M \times M \\
\downarrow \\
M
\end{array}
\]

commute.

Here "$\text{id}_M : \{*\} \times M \to M" is the bijection $(\ast, a) \mapsto a$ for all $a \in M$.

**Definition** A monad on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \to \mathcal{C}$ (an endo-functor) together with two natural transformations: the unit $\eta : \text{id}_\mathcal{C} \Rightarrow T$ and the multiplication $\mu : T \circ T \Rightarrow T$, so that the following three diagrams commute in the functor category $[\mathcal{C}, \mathcal{C}]$.
Note that since \( \text{id}_e \cdot T = T \), \( \text{id}_T : \text{id}_e \cdot T \Rightarrow T \) makes sense. For the same reason \( \text{id}_T : T \Rightarrow \text{id}_e \) makes sense as well.

**Remark.** Note also that the three diagrams above for natural transformations amount to:

\[
\begin{array}{ccc}
Tc & \xrightarrow{Tc} & T^2 c \\
\downarrow M_T & = & \downarrow M_T \\
Tc & \xrightarrow{M_T} & T^2 c
\end{array}
\]

and

\[
\begin{array}{ccc}
T^3 c & \xrightarrow{T^3 c} & T^2 c \\
\downarrow M_T & = & \downarrow M_T \\
T^2 c & \xrightarrow{M_T} & T^3 c
\end{array}
\]

commute in the category \( C \).

**Example (the writer monad).** Fix a monoid \((M, \cdot, e)\). Define a functor \( T : \text{Set} \rightarrow \text{Set} \) by

\[
T(X, Y) = X \times M \xrightarrow{f \times \text{id}_M} Y \times M.
\]

The function \( f \times \text{id}_M \) defines a natural transformation \( \eta : \text{id}_{\text{Set}} \Rightarrow T \) by \( \eta_X : X \xrightarrow{(f \times \text{id}_M) x} X \times M \), i.e. \( \eta_X(x) = (x, e) \) for all \( x \in X \). \( \eta \) is a natural transformation since for any function \( f : X \rightarrow Y \) the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta_X & & \downarrow \eta_Y \\
X \times M & \xrightarrow{f \times \text{id}_M} & Y \times M
\end{array}
\]

commutes.

The multiplication \( \mu : T^2 \Rightarrow T \) is defined by setting

\[
\mu_X : X \times M \times M \rightarrow X \times M
\]

to be \( \mu_X(x, a, b) = (x, a \cdot b) \). It is easy to see that \( \mu \) is a natural transformation.

We now check the commutativity of the three diagrams.

For any set \( X \), \( (\mu_X \circ \eta_T X) : TX \rightarrow TX \) is given by \((x, a) \rightarrow (x, a, e) \rightarrow (x, a \cdot e)\).

Since \( a \cdot e = a \) \( \forall a \in M \), \( \mu_X \circ \eta_T X = \text{id}_{TX} \) for all \( X \in \text{Set} \).

Since for any morphism \( f \)

\[
(T(f))(x, a) = ((x, e), a) \quad \forall (x, a) \in TX = X \times M.
\]

\[
\Rightarrow (\mu_X \circ T(\eta_X))(x, a) = M_X((x, e), a) = (x, ea) = \omega_{TX}(x, e).
\]

Finally, \( T(\mu_X) : M_X \times \text{id}_M : X \times M \times M \times M \rightarrow X \times M \times M \) \((x, a, b, c) \rightarrow (x, a \cdot b, c)\)

while \( \mu_{TX} : (X \times M) \times M \times M \rightarrow (X \times M) \times M \) \((a, b, c) \rightarrow (a, b \cdot c)\)

Hence \( (\mu_X \circ \mu_{TX})(x, a, b, c) = (x, a(b(c)) \) while
$(m_x \circ T^m_x)(x, a, b, c) = \land_x (x, a, b, c) = (x, (ab)c)$. So the associativity of $\mu$ follows from the associativity of the multiplication in $M$.

Many monads come from "obvious" adjunctions by way of the following

**Theorem 36.1** Let $\mathcal{E} \xleftarrow{F} \mathcal{B}$ be an adjunction with the $\eta: \text{id}_{\mathcal{E}} \Rightarrow GF$ and counit $\varepsilon: FG \Rightarrow \text{id}_{\mathcal{B}}$. Then $(T = G \circ F, \eta: \text{id}_{\mathcal{E}} \Rightarrow T, \mu = G \circ F: T^2 \Rightarrow T)$ is a monad on $\mathcal{E}$.

**Proof:** next lecture.

**Example** The finite list monad. Consider the free/forgetful adjunction $F: \text{Set} \leftarrow \text{Mon}: U$, $T = UF: \text{Set} \rightarrow \text{Set}$ sends a set $X$ to the set $U(X^*)$ underlying the monoid $X^*$ of finite lists of elements of $X$. It will be convenient to suppress the functor $U$ and denote by $X^*$ both the monoid and its underlying set.

Then the components of the unit $\eta: \text{id}_{\mathcal{E}} \Rightarrow UF$ of the adjunction (and of the monad) are functions $\eta_X: X \rightarrow X^*$, $\eta_X(x) = (x)$, where $(x)$ is a one element list.

The component of the counit $\varepsilon: FU \Rightarrow \text{id}_{\text{Mon}}$ at a monoid $M$ is the homomorphism $\varepsilon_m: M^* \rightarrow M$, $\varepsilon_m((m_1, \ldots, m_n)) = m_1 \cdot m_2 \cdots m_n$. The multiplication in $M$ of the elements of the list $(m_1, \ldots, m_n)$. It follows that the $X$-component $\mu_X: (X^*)^* \rightarrow X^*$ of the monad multiplication $\mu: T^2 \Rightarrow T$ is defined by

$$\mu_X((x_{11}, x_{12}, \ldots, x_{1n}), (x_{21}, x_{22}, \ldots, x_{2n}), \ldots, (x_{k1}, \ldots, x_{kn})) = (x_{11}, x_{21}, x_{21}, \ldots, x_{kn}).$$

In other words the components of $\mu$ "flatten" a list of lists into one list.

**Example (the maybe monad).** The maybe monad on Set is defined as follows.

The functor $T: \text{Set} \rightarrow \text{Set}$ adjoins to a set $X$ a one point set $X^*: T(X) = X \cup \{x\}$.

The maybe monad arises from the adjunction $\text{Set} \xleftarrow{F} \text{Set}_X$, where $\text{Set}_X$ is the category of pointed sets, $U: \text{Set}_X \rightarrow \text{Set}$ in the forgetful functor,
\( U((X,x)) = X. \) Its left adjoint \( F \) sends a set \( X \) to the pair \( (X \sqcup x, x) \).

An \( X \)-component of the unit \( \eta : \text{id}_{\text{set}} \Rightarrow T \) is given by \( \eta_X : X \to X \sqcup x, x \), which is the inclusion \( X \hookrightarrow X \sqcup x, x \). The monad multiplication \( \mu : T \to T \) is defined by sending a set with two 1-element sets adjoint to the set with a 1-element set adjoint: \( \mu_X : X \sqcup x, x \sqcup x \to X \sqcup x, x \). \( \mu_X \) in identity on \( X \sqcup x, x \) and sends the "extra" point \( x \) to \( x \).

**Remark.** The notion of a monad is not just analogous to the notion of a monoid.

Monads and ordinary monoids are both "monoids in a monoidal category."

Roughly speaking, a monoidal category \( \mathcal{C} \) in a category with a functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \), monoidal product, which is associative, and an object \( I \in \mathcal{C} \), which is a left and right unit for \( \otimes \).

The category \( \text{Set} \) can be made into a monoidal category by choosing \( \otimes : \text{Set} \times \text{Set} \to \text{Set} \) to be the Cartesian product (there are other choices): \( (X \to Y) \otimes (Z \to W) = X \times Z \xrightarrow{f \times g} Y \times W \).

The unit \( I \) is a one element set \(*\): for any set \( X \), we have bijections \( \lambda_X : \{x\} \times X \to X \), \( \mu_X : X \times \{x\} \to X \) which are natural in \( X \).

For any category \( \mathcal{C} \), the functor category \([\mathcal{C}, \mathcal{C}]\) is monoidal. The monoidal product \( \otimes \) is given by the (horizontal) composition:

\[
(\xi \otimes \eta) \\
= (\xi \otimes \eta) \circ (\xi \otimes \eta) \\
= \xi \circ (\eta \circ \eta) \\
= \xi \circ (\eta \circ \eta)
\]

A monoid in a monoidal category \((\mathcal{C}, \otimes, I)\) is an object \( m \in \mathcal{C} \) together with a multiplication \( \mu : m \otimes m \to m \) and a unit \( \eta : I \to m \) so that

\[
\begin{align*}
\mu \circ \mu & \to m, \\
\mu \circ \id & \to m, \\
\eta \circ \id & \to m
\end{align*}
\]

commute.