Last time: Finished showing that the three views of adjunctions — as natural bijections between homs, as a functor + universal arrows and as a pair of natural transformations (the unit and the counit) — satisfying the triangle identities — are all equivalent and revisited some examples:

• left adjoint to \( U: \text{Mon} \to \text{Set} \) (which is the free monoid functor)
• left adjoint to \( \Delta: \mathcal{C} \to [I, \mathcal{C}] \) (the colimit functor)

Example: right adjoint to \( \Delta: \mathcal{C} \to [I, \mathcal{C}] \). Suppose every functor \( F: I \to \mathcal{C} \) has a limit. For each \( F \in [I, \mathcal{C}] \) choose \( \lim F \in \mathcal{C} \). We then have a natural transformation \( \psi_F: \Delta(\lim F) \Rightarrow F \) and the pair \( \{ \lim F, \psi_F \} \) has the following (co)universal property:

\[
\forall c \in \mathcal{C}, \; \exists! \; \alpha: c \to \lim F \text{ so that } \Delta(\lim F) \xrightarrow{\psi_F} F \xleftarrow{\alpha} \Delta(c)
\]

commutes in \([I, \mathcal{C}]\).

Consequently, the function \( F \mapsto \lim F \) extends to a functor \( \lim: [I, \mathcal{C}] \to \mathcal{C} \) which is right adjoint to \( \Delta: \mathcal{C} \to [I, \mathcal{C}] \).

The \( F \)-component \( (F \circ \Delta c) \) of the counit of adjunction

\[
\varepsilon : \Delta \circ \lim \Rightarrow \text{id}_{[I, \mathcal{C}]} \circ \Pi_F : \Delta(\lim F) \Rightarrow F.
\]

The unit \( \eta : \text{id}_{\mathcal{C}} \Rightarrow \lim \circ \Delta \) assigns to each \( c \in \mathcal{C} \), \( \text{id}_c : c \to c \), provided we choose \( \lim (\Delta c) \) to be \( c \) (\( \forall c \in \mathcal{C} \)).

Our next result is that adjunctions compose.

**Lemma 34.1** Suppose \( A \xleftarrow{G} B \xrightarrow{H} C \). Then \( A \xleftarrow{G} \xrightarrow{HK} C \).

**Proof** Let \( \theta: \text{Hom}_B(H(-), -) \Rightarrow \text{Hom}_A(\text{-}, K(-)) \) and

\[
\tau: \text{Hom}_B(F(-), -) \Rightarrow \text{Hom}_A(\text{-}, G(-))
\]

denote the natural isomorphisms.
34.2

Then $\forall \ c \xrightarrow{l} c' \ in \ C, \ \forall \ a' \xrightarrow{m} a \ in \ A$ the squares

\[
\begin{array}{ccc}
\text{Hom}_C(HFa, c) & \xrightarrow{\theta_{Fa,c}} & \text{Hom}_B(Fa, Kc) \\
HF(m)^* \circ l_x & \downarrow & \text{Flm}^* \circ (K)_x \\
\text{Hom}_C(HFa', c) & \xrightarrow{\theta_{Fa',c}} & \text{Hom}_B(Fa', Kc') \\
\end{array}
\]

commute. Hence the outer square in the diagram

\[
\begin{array}{ccc}
\text{Hom}_C(HFa, c) & \xrightarrow{\theta_{Fa,c}} & \text{Hom}_B(Fa, Kc) \\
HF(m)^* \circ l_x & \downarrow & \text{Flm}^* \circ (K)_x \\
\text{Hom}_C(HFa', c) & \xrightarrow{\theta_{Fa',c}} & \text{Hom}_B(Fa', Kc') \\
\end{array}
\]

commutes as well. Therefore $\tau \circ \Theta : \text{Hom}_C(HF(-), \cdot) \Rightarrow \text{Hom}_A(-, G(K(-)))$ is a natural isomorphism, i.e. $HF \Rightarrow GK$.

\[\blacksquare\]

**RAPL:** Right Adjoint Preserves Limits.

**Theorem 34.2** Suppose $A \xleftarrow{\eta} B$. Then $G$ preserves all the limits that exist in $B$.

If $D : I \to B$ is a functor with a limit cone $(L, \{\lambda_j : L \to D(j)\}_{j \in I})$ then $(G(L), \{G\lambda_j : G(L) \to GD(j) \}_{j \in I})$ is a limit cone in $A$.

**Proof** Suppose $(a, \{a_i : a \to G(D(i))\}_{i \in I})$ is a cone in $A$ over $GD$. Then $\forall \ i \xrightarrow{\alpha_i} a_i$ in $D$, $a \xrightarrow{\alpha_i} GD(i)$ commutes. \Rightarrow $(GD(r)) \xrightarrow{\alpha_i} GD(r(i))$.

Since $F \Rightarrow G$, the diagram

\[
\begin{array}{ccc}
\text{Hom}_A(a, G(D(i))) & \xrightarrow{\text{Hom}_A(\alpha_i)} & \text{Hom}_B(Fa, D(i)) \\
\downarrow & & \downarrow \\
\text{Hom}_A(a, G(D(i'))) & \xrightarrow{\text{Hom}_A(\alpha_i')} & \text{Hom}_B(Fa, D(i'))
\end{array}
\]

commutes.

\[
\Rightarrow D(i) \xrightarrow{\alpha_i} D(i') \Rightarrow (Fa, \{\alpha_i : Fa \to D(i)\}_{i \in I}) \text{ in a cone over } D.
\]
\[ F \alpha \to F \beta \text{ so that } \lambda_i \circ F \beta = \alpha_i \text{ i.e. } \varphi_i^{F(\beta)} \text{ commutes } \forall i \in I. \]

Since \[ - : \text{Hom}_A (a, G(c)) \to \text{Hom}_B (F(a), L) \] is a bijection \( \exists ! h : a \to G(c) \text{ st } \overline{h} = \beta. \)

Since \[ \text{Hom}_A (a, G(c)) \to \text{Hom}_B (F(a), L) \]
\[ \downarrow (G \lambda_i)_x \downarrow (\lambda_i)_x \]
\[ \text{Hom}_A (a, G(D(i))) \to \text{Hom}_A (F(a), D(i)) \] commutes for all \( i \in I, \)
\[ (G \lambda_i)_x h = (\lambda_i)_x \overline{h} = \lambda_i \circ F \beta = \alpha_i. \]
Since \( - \) is a bijection \( \alpha_i = (G \lambda_i)_x h = G \lambda_i \circ h. \)

Thus \( a \maps h \to G(L) \) is the unique morphism in \( A \) so that \( \alpha_i \maps (G \lambda_i)_x \to G(\lambda_i) \) commutes \( \forall i. \)

\[ (G(L), (G \lambda_i)_x; e_x) \text{ is a limit cone of } G \circ D. \]

**Example** The forgetful functor \( U : \text{Mon} \to \text{Set} \) has a left adjoint \( F : \text{Set} \to \text{Mon}, \) \( F(X) = X^* \). Since \( F \) is a left adjoint \( F \) preserves colimits and, in particular, coproducts. Coproducts in \( \text{Set} \) are disjoint unions. Hence for any two sets \( X, Y \) the free monoid \( F(X \sqcup Y) \) is the coproduct, in the category \( \text{Mon} \) of monoids of the free monoids \( F(X) \) and \( F(Y) \). In particular the coproduct of \( F(X) \) and \( F(Y) \) exists in \( \text{Mon} \) (and equals \( F(X \sqcup Y) \)).

On the other hand, since \( U \) is right adjoint to \( F \), \( U \) preserves limits and, in particular, products. This is why if \( M, N \) are two monoids then the set underlying their product has to be \( U(M) \times U(N) \).

**Corollary 34.3** Left adjoints preserve colimits.

**Proof.** Duality.
Example The forgetful functor \( U : \text{Mon} \to \text{Set} \) has no right adjoint. If it did, it would preserve all colimits and, in particular, initial objects. But the initial object in \( \text{Mon} \) is a one-element monoid set, while the initial object in \( \text{Set} \) is the empty set.

Example \( U : \text{Vect} \to \text{Set} \) does not preserve coproducts:

The initial vector space is the zero-dimensional vector space \( 0_\text{d} \).
\[ U(0_\text{d}) = 0_\text{d}, \] which is not empty, hence is not initial in \( \text{Set} \). Hence \( U : \text{Vect} \to \text{Set} \) has no right adjoint.

Example The forgetful functor \( U : \text{Top} \to \text{Set} \) has a right adjoint and a left adjoint. Consequently \( U \) preserves both limits and colimits. This is why the set underlying a product of topological spaces has to be the product of the underlying sets.

Similarly, the set underlying a coproduct of topological spaces is the disjoint union of the underlying sets.