A functor $G : B \to A$ has a left adjoint $F : A \to B \iff$

there is a universal arrow from $a$ to $G$ (initial object of $(a \downarrow G)$).

Dually, a functor $F : A \to B$ has a right adjoint $\iff$

there is a couniversal arrow from $F$ to $b$ (terminal object of $(F \downarrow b)$).

Today, “uniqueness” of left and right adjoint to a given functor.

Adjunction in terms of unit and counit.

**Lemma 32.1** Left adjoints are unique up to isomorphism: suppose $G : B \to A$,

$F_1, F_2 : A \to B$ are functors with $F_1 \dashv G$ and $F_2 \dashv G$. Then there is a

natural isomorphism $\alpha : F_1 \cong F_2$.

**Proof** $\forall a \in A, b \in B$ we have bijections

$\text{Hom}_B (F_1 a, b) \cong \text{Hom}_A (a, GB) \cong \text{Hom}_B (F_2 a, b)$

that are natural in $a$ and $b$. Hence we have a family of bijections

$\Gamma_{a,b} : \text{Hom}_B (F_1 a, b) \to \text{Hom}_B (F_2 a, b)$, natural in $a$ and $b$.

**Naturality in $b$** says that $\forall b' \in B$ and $a \in A$ the diagram

$\text{Hom}_B (F_1 a, b) \xrightarrow{\Gamma_{a,b}} \text{Hom}_B (F_2 a, b)$

$
\downarrow h_x \quad \quad \downarrow h_x
$

commutes, i.e

$\text{Hom}_B (F_2 a, b') \xrightarrow{\Gamma_{a,b'}} \text{Hom}_B (F_2 a, b')$

$\Rightarrow$ For a fixed $a \in A$, $\{ \Gamma_{a,b} \}_{b \in B}$ are components of a natural isomorphism

$\Gamma_a : \text{Hom}_B (F_1 a, -) \cong \text{Hom}_B (F_2 a, -)$.

Recall that the *Yoneda embedding* $y^* : B^{op} \to \text{[B, Set]}$ is defined by

$y^* (b \xrightarrow{b'} b') = \text{Hom}_B (b',-) \xrightarrow{b} \text{Hom}_B (b, -)$ and that $y^*$ is fully faithful. Hence

$\exists \alpha_a : F_2 a \to F_1 a$ so that

$y^*( F_2 a \xrightarrow{\alpha_a} F_1 a ) = y^* (F_1 a) \xrightarrow{\Gamma_a} y^* (F_2 a)$

We need to check: $\{ \alpha_a \}_{a \in A}$ are components of $\alpha : F_2 \Rightarrow F_1$, i.e. $\forall k : a \to a'$
Since \( y^*: \mathcal{B}^{\text{op}} \to \mathcal{B}, \mathcal{S}^{\text{et}} \) is fully faithful, it's enough to check that
\[
\eta_{a}^* (F_a(a)) \Rightarrow \eta_{a}^* (F_{a}(a))
\]
\[
\forall b: \mathcal{B}, (y^*(F_{a}(a))(b) = \text{Hom}_{\mathcal{B}}(F_{a}(a), b)
\]
\[
y^*(F_k)_{b}: \text{Hom}_{\mathcal{B}}(F_{a}(a), b) \to \text{Hom}_{\mathcal{B}}(F_{a}(a), b) \text{ is } (F_k)^* \quad (\tau_a\cdot)_{b} = \tau_{a,b} \text{ etc.}
\]
So commutativity of (a) is equivalent to: \( \forall b: \mathcal{B} \)
\[
\text{Hom}_{\mathcal{B}}(F_{a}(a), b) \xrightarrow{\tau_{a,b}} \text{Hom}_{\mathcal{B}}(F_{a}(a), b)
\]
\[
(F_k)^* \uparrow \quad \uparrow (F_k)^*
\]
\[
\text{Hom}_{\mathcal{B}}(F_{a}(a), b) \xrightarrow{\tau_{a,b}} \text{Hom}_{\mathcal{B}}(F_{a}(a), b) \quad \text{commutes.}
\]
But this is naturality of \( \tau_{a,b} \) in \( a \). Hence (a) commutes \( \forall b: \mathcal{B} \)
and therefore we have a natural isomorphism \( d: F_1 \Rightarrow F_2 \).  

**Corollary 32.2** Right adjoints are unique up to isomorphism: if \( F \dashv G_1 \) and \( F \dashv G_2 \) then \( G_1 \) is naturally isomorphic to \( G_2 \).

**Proof** By duality \( F \dashv G_1, F \dashv G_2 \Rightarrow G_1^{\text{op}} \dashv F^{\text{op}} \) and \( G_2^{\text{op}} \dashv F^{\text{op}} \). By Lemma 32.1 \( G_1^{\text{op}} \) is naturally isomorphic to \( G_2^{\text{op}} \). Hence \( G_1 \) is naturally isomorphic to \( G_2 \).  

**Triangle identities**

Let \( \Theta: \text{Hom}_{\mathcal{B}}(-, \cdot) \Rightarrow \text{Hom}_{\mathcal{A}}(-, \cdot) \) be an adjunction. Recall that we have a pair of natural transformations \( \eta: \Theta_{A} \Rightarrow GF, \varepsilon: FG \Rightarrow \text{id}_{\mathcal{B}} \)
which are defined by
\[
\eta_a = \Theta_{F(a)}(\text{id}_{F(a)}) \equiv \text{id}_{F(a)} \quad \text{and} \quad \varepsilon_b = (\Theta_{\text{G(b)}})^{-1}(\text{id}_{\text{G(b)}}) \equiv \text{id}_{\text{G(b)}}
\]
for all \(a \in A, b \in B\). Here as before \(-\) denotes either the bijection
\[
\Theta_{a,b} : \text{Hom}(Fa, b) \to \text{Hom}(a, GB)
\]
or its inverse.

Theorem 32.2 (triangle identities) Let \(A \xleftarrow{G} B\) be an adjunction, \(\eta : \text{id}_A \Rightarrow GF\), \(\varepsilon : FG \Rightarrow \text{id}_B\) the corresponding unit and counit, respectively. Then the diagrams
\[
\begin{align*}
GFG & \xrightarrow{\varepsilon FG} FG \\
(32.3) & \quad \begin{array}{c}
\varepsilon \downarrow \quad \downarrow \\
G & \quad G
\end{array}
\end{align*}
\]
and
\[
\begin{align*}
FGF & \xrightarrow{G \varepsilon F} FGF \\
(32.4) & \quad \begin{array}{c}
\varepsilon \downarrow \quad \downarrow \\
F & \quad F
\end{array}
\end{align*}
\]
commute in the functor categories \([B, A]\) and \([A, B]\), respectively.

Remark 1 Recall that \(\forall b \in B, (\eta G)_b = \eta_{G(b)}\) and \((G \varepsilon)_b = G(\varepsilon_b)\) (see Lecture 27). So (32.3) says that \(\forall b \in B, \quad \text{id}_{G(b)} = G(\varepsilon_b) \circ \eta_{G(b)}\), while (32.4) says that \(\forall a \in A\)
\[
\text{id}_{Fa} = \varepsilon F(a) \circ G(\varepsilon_a).
\]

Proof Since \(\Theta_{a,b}\)'s are bijections natural in \(a\) and \(b\), \(\forall a \in A, b \in B \quad \forall f : Fa \to b\)
The diagram
\[
\begin{array}{ccc}
\text{Hom}_A(Fa, Fa) & \xrightarrow{\Theta_{a,b}} & \text{Hom}_B(a, GFa) \\
\downarrow f_\star & & \downarrow (Gf)_\star \\
\text{Hom}_A(Fa, b) & \xrightarrow{\Theta_{a,b}} & \text{Hom}_B(a, GB)
\end{array}
\]
commutes.
\[
\Rightarrow \quad \overline{f} = \Theta_{a,b}(f) = \Theta_{a,b}(f_\star(\text{id}_{Fa})) = (Gf)_\star \left(\Theta_{a,b}(\text{id}_{Fa}) \right) = Gf \circ \varepsilon_a.
\]
So \(\overline{f} = G(f) \circ \varepsilon_a\) \(\forall f : Fa \to b \in \text{Hom}_B(Fa, b)\)

Similarly \(\forall g : a \to GB \in \text{Hom}_B(a, GB)\) the diagram
\[
\begin{array}{ccc}
\text{Hom}_A(FGB, b) & \xleftarrow{\Theta_{a,b}^{-1}} & \text{Hom}_A(Gb, GB) \\
(Fg)^* \downarrow & & \downarrow g^* \\
\text{Hom}_A(Fa, b) & \xleftarrow{\Theta_{a,b}^{-1}} & \text{Hom}_A(a, GB)
\end{array}
\]
commutes.
\[
\Rightarrow \quad \overline{g} = \Theta_{a,b}^{-1}(g) = g^* (\text{id}_{Gb}) = (Fg)^* (\text{id}_{Gb}) = (Fg)^* \varepsilon_b = \varepsilon_b \circ Fg.
\]
\[
\Rightarrow \quad (2) \quad \overline{g} = \varepsilon_b \circ Fg \quad \text{for all } g : a \to GB.
\]
Now \( \forall b \in B \) equation (a) above \( \def \ref {32.4} \)
\[
(G \circ \eta) \circ \varepsilon = (G \circ \eta) \circ \varepsilon \circ \eta \circ \varepsilon = \eta \circ \varepsilon \circ \eta = \id_B. \] 
This proves \( (32.3) \).

Similarly, \( \forall a \in A \) equation (a)
\[
(\varepsilon F \circ \varepsilon) \circ \varepsilon = (\eta a) \circ \varepsilon \circ \eta a = \eta a \circ \id_B = \id_B. \] 
This proves \( (32.4) \).

Theorem 32.2 has a converse, which we’ll prove in the next lecture.

**Theorem 32.5** Suppose \( F : A \rightarrow \beta : G \) are two functors, \( \eta : \id_A \Rightarrow GF, \varepsilon : FG \Rightarrow \id_B \)
+ two natural transformations that satisfy the triangle identities \( (32.3) \) and \( (32.4) \). Then there exists a natural isomorphism \( \Theta : \text{Hom}_B(F -, G -) \Rightarrow \text{Hom}_A(-, G \circ -) \).
+ In other words the data \( (F, G, \varepsilon, \eta) \) determine an adjunction \( (F, G, \Theta) \).

**Remark** Together the two theorems \( (32.2 \text{ and } 32.5) \) say that we can define
+ an adjunction either as a triple \( (F, G, \Theta : \text{Hom}_B(F -, G -) \Rightarrow \text{Hom}_A(-, G \circ -)) \) with \( \Theta \)
+ a natural isomorphism or as a quadruple \( (F, G, \eta : \id_A \Rightarrow GF, \varepsilon : FG \Rightarrow \id_B) \)
+ which is subject to the triangle identities \( (32.3) \) and \( (32.4) \).