Pushouts

Consider the category $X$ with 3 objects and two nonidentity morphisms.

A pushout in a category $C$ is the colimit of a functor $F: \left( \frac{X^2}{x \to y} \right) \to C$.

Equivalently, given 3 objects and two morphisms $f \to a$ in $C$ the pushout of this diagram is an object $d$ of $C$ together with a pair of morphisms $b \to d$, $a \to d$ so that given any object $e$ of $C$ and a pair of morphisms $b \to e$, $a \to e$ so that $f \downarrow a \to c \to b$ commutes, $f$ unique morphism $d \to e$ so that $c \to b$ commutes.

Example Since a pushout is a colimit, and since $\text{Set}$ has colimits, pushouts exist in $\text{Set}$. Explicitly, given 3 sets and two functions $f : C \to B$, the pushout in the quotient $(A \cup B) / \sim$ of the disjoint union by the equivalence relation generated by: $a \sim b$ for $a \in A$, $b \in B$ if $f(a) = f(b)$ so that $a = f(c)$ and $b = g(c)$. The structure maps $\eta_A : A \to (A \cup B) / \sim$, $\eta_B : B \to (A \cup B) / \sim$ are the inclusions $A \hookrightarrow A \cup B$, $B \hookrightarrow A \cup B$ followed by the quotient map $q : A \cup B \to (A \cup B) / \sim$.

Since $\text{Top}$ is cocomplete it has pushouts. They can be constructed as follows: let $A \leftarrow B \rightarrow C$ be a triple of topological spaces with two continuous functions. As in the case of sets, the pushout of this diagram is the quotient $(A \cup C) / \sim$. 

Last time The categories $\text{Vec}$, $\text{Top}$ have coequalizers and consequently are cocomplete.
of the coproduct $A \sqcup C$ (in $\text{Top}$) by the equivalence relation $\sim$ generate by

$$S = \{ (a, c) \mid \exists b \in B \text{ with } a = f(b), c = g(c) \}.$$  

As in the case of $\text{Set}$ the structure maps $\iota_A : A \to (A \sqcup C)/\sim$ and $\iota_C : (A \sqcup C)/\sim$ are the composites of the "inclusions" $A \hookleftarrow A \sqcup C$, $C \hookleftarrow A \sqcup C$ followed by the quotient map $q : A \sqcup C \to (A \sqcup C)/\sim$.

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**Natural transformations**

**Definition.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors between two categories. A natural transformation $\alpha$ from $F$ to $G$ assigns to each object $c$ of $\mathcal{C}$ a morphism $\alpha_c : F(c) \to G(c)$ of $\mathcal{D}$ so that for any morphism $c \to c'$ of $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\alpha_c} & G(c) \\
\downarrow F(c) \to c' & & \downarrow G(c) \to c'
\end{array}
$$

commutes. We write $\alpha : F \Rightarrow G$ and $\alpha : \mathcal{C} \to \mathcal{D}$.

Note $\alpha$ is a map from $\mathcal{C}$ to $\mathcal{D}$: $\alpha : \mathcal{C} \to \mathcal{D}$, $c \mapsto \alpha_c$.

**Example.** Natural transformations and cones. Let $\mathcal{C}, \mathcal{D}$ be two categories. For any object $d$ of $\mathcal{D}$ we have a constant functor $\Delta_d : \mathcal{C} \to \mathcal{D}$. It is defined by

$$\Delta_d(c \to c') = d \quad \text{for all morphisms } c \to c' \text{ of } \mathcal{C}.$$  

For any functor $F : \mathcal{C} \to \mathcal{D}$ a natural transformation $\alpha : \Delta_d \Rightarrow F$ assigns to each object $c$ of $\mathcal{C}$ a morphism $\alpha_c : \Delta_d(c) = d \to F(c)$ of $\mathcal{D}$ so that for any morphism $c \to c'$ the diagram

$$
\begin{array}{ccc}
\Delta_d(c) & \xrightarrow{\alpha_c} & F(c) \\
\downarrow d \to c' & & \downarrow F(c) \to c'
\end{array}
$$

commutes.

Thus $\alpha : \Delta_d \Rightarrow F$ "is" a cone on $F$.

**Example.** Recall that we have a contravariant functor $\left(\cdot \right)^\ast : \text{Vec}_{IR} \to \left(\text{Vec}_{IR}\right)^{\text{op}}$, 

$$
(V \hookleftarrow W) \mapsto (W \xrightarrow{\leftarrow} V^\ast)
$$
where \( W^* = \text{Hom}(W, R) \), \( V^* = \text{Hom}(V, R) \) and \( (T^*)(l) = \text{lo} T \), \( \forall \ l \in W \).

Applying the functor \( (\cdot)^* \) again we get \( \cdot^*: (\text{Vect}_R)^{op} \to ((\text{Vect}_R)^{op})^{op} \).

On the other hand for any vector space \( V \) there is a linear map \( \text{ev}_V: V \to (V^*)^* \), which is defined by \( (\text{ev}_V(w))(l) = \langle w, l \rangle \) for all \( v \in V \) and \( l \in V^* \).

These linear maps assemble into a natural transformation \( \text{ev}_V: \text{id}_{\text{Vect}_R} \Rightarrow (\cdot)^*: (\text{Vect}_R)^{op} \to ((\text{Vect}_R)^{op})^{op} \).

For any linear map \( T: V \to W \) the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\text{ev}_V} & (V^*)^* \\
\downarrow T & & \downarrow (T^*)^* \\
W & \xrightarrow{\text{ev}_W} & (W^*)^*
\end{array}
\]

commutes.

This is because \( \forall l \in W^*, \forall v \in V, \)

\[
((T^*)^* \text{ev}_V(w))(l) = (\text{ev}_V(w))(T^*l) = (T^*l)(v) = l(Tv) = (\text{ev}_W(Tv))(l).
\]

It is in this sense that taking double duals is "natural."

**Definition.** Let \( E, D \) be two categories, \( F, G, H: E \to D \) two functors and \( \alpha: F \Rightarrow G, \beta: G \Rightarrow H \) two natural transformations.

The **vertical composition** of \( \beta \) and \( \alpha \) is the transformation

\[
\beta \circ \alpha: F \Rightarrow H,
\]

which is defined by \( (\beta \circ \alpha)_c = \beta_c \circ \alpha_c \) for all \( c \in E \).

Note that \( \beta \circ \alpha \) is a natural transformation. This because \( \forall c \in E \),

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\alpha_c} & G(c) \\
\downarrow F(v) & & \downarrow G(v) \\
F(c') & \xrightarrow{\alpha_{c'}} & G(c')
\end{array}
\]

the left and the right squares commute.

\[
\begin{array}{ccc}
F(c) & \xrightarrow{\beta_c \circ \alpha_c} & H(c) \\
\downarrow F(v) & & \downarrow H(v) \\
F(c') & \xrightarrow{\beta_{c'} \circ \alpha_{c'}} & H(c')
\end{array}
\]

Hence the outer square \( \begin{array}{ccc} F(v) & \xrightarrow{} & H(v) \\
\downarrow F(c') & & \downarrow H(c) \\
\end{array} \) commutes as well.
Exercise

Check that vertical composition is associative. That is, given four functors $F, G, H, K : C \to D$ and three natural transformations $\alpha : G \Rightarrow H, \beta : F \Rightarrow G, \gamma : K \Rightarrow L$, show that

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

Hint: Compute the components $((\gamma \circ \beta) \circ \alpha)_c$ and $(\gamma \circ (\beta \circ \alpha))_c$ at some object $c$ of $C$, and use the fact that composition of morphisms in $D$ is associative.

Remark. The horizontal composition of natural transformations occurs when we have three categories, 4 functors and two natural transformations like this:

$\begin{array}{ccc}
A & \xrightarrow{H} & B \\
\Downarrow & & \Downarrow \\
C & \xrightarrow{K} & D
\end{array}$

The horizontal composition $\beta \ast \alpha$ is then a natural transformation from $K \circ G$ to $L \circ H$. We'll define it later.

Functor categories

Fix two categories $C$ and $D$. The functor category $[C, D]$ is defined as follows:

- the objects of $[C, D]$ are functors $F : C \to D$.
- the morphisms are natural transformations.
- the composition of morphisms is the vertical composition.
- the unit/identity morphism $1_F$ on a functor $F$ is defined by $(1_F)_c = 1_{F(c)}$ for all $c \in C$. Here as before $1_{F(c)} : F(c) \to F(c)$ is the identity morphism in the target category $D$.

Another common notation for $[C, D]$ is $D^C$. This is by analogy with $X^Y$ = the set of maps from a set $Y$ to a set $X$.

Functor categories are important and sometimes occur rather unexpectedly. For example, a directed graph is a functor and the category of directed graphs is a functor category. We'll discuss graphs in the next lecture.