Last time. Limits of functors. A limit of a functor $F : I \to \mathcal{C}$ in an object $L$ of $\mathcal{C}$ and a collection of morphisms $\{ \pi_i : L \to F(i) \}_{i \in I_0}$ so that
\[
\forall i, j \in I \quad \pi_i \circ F(f) = \pi_j \circ F(f)
\]
commutes, i.e. $(L, (\pi_i : L \to F(i))_{i \in I_0})$ is a cone over $F$.

and the pair $(L, (\pi_i : L \to F(i))_{i \in I_0})$ has the appropriate universal property:
\[
\forall c \in \mathcal{C} \text{ and any set of morphisms } \{ f_i : c \to F(i) \}_{i \in I_0} \text{ so that } \forall i, j \in I
\]
\[
\pi_i \circ f_i = \pi_j \circ f_j
\]
commutes, i.e., $(c, \{ f_i : c \to F(i) \}_{i \in I_0})$ is a cone on $F$.

\[\exists! \, f : c \to L \text{ such that } \pi_i \circ f_i = \pi_i \circ f \text{ commutes } \forall i \in I_0.\]

More abstractly a limit of a functor $F : I \to \mathcal{C}$ is a terminal object in the category $\text{Cone}(F)$ of cones on $F$ (and so the limit $\text{lim } F$ is unique up to a unique isomorphism in $\text{Cone}(F)$).

Special case! $I$ is discrete (i.e. I only has the identity morphisms). Then $\text{lim } (F : I \to \mathcal{C})$ is the product $\prod_{i \in I_0} F(i)$ of the collection $\{ F(i) \}_{i \in I_0}$.

Another special case: we have a category $\mathcal{C}$, $a, b \in \mathcal{C}_0$, $f, g : a \to b$ two morphisms. Then $I = \{ a \xrightarrow{a} b \}$ is a subcategory of $\mathcal{C}$ which comes with the inclusion functor $\iota : I \to \mathcal{C}$. The limit of $F$ (when it exists) is, by definition the equalizer of $a \xrightarrow{f} b$ : it's an object $e \in \mathcal{C}$, a morphism $e \xrightarrow{e} a$ so that $f \circ e = g \circ e$ which has the following universal property:
\[\exists! \, f : c \to a \text{ such that } f \circ e = g \circ e.\]

Exercise An equalizer $e \xrightarrow{e} a$ of $a \xrightarrow{f} b$ is monic.
Definition Let \( \mathcal{C} \) be a category, \( a \xrightarrow{f} b \xrightarrow{g} c \) three objects and two morphisms.

These data define a subcategory \( I = \{ a \xrightarrow{f} b \xrightarrow{g} c \} \) of \( \mathcal{C} \) and the inclusion functor \( F : I \to \mathcal{C} \).

The limit of \( F \) (when it exists) is called the fiber product of \( a \xrightarrow{f} b \xrightarrow{g} c \).

In more detail, it's an object \( axb \in \mathcal{C} \) and three morphisms

\[
\pi_a : axb \to a, \quad \pi_b : axb \to b, \quad \pi_c : axb \to c
\]

so that \( \pi_c \circ \pi_a = \pi_b \circ \pi_a \) and the appropriate universal property holds. Note that since \( \pi_b = f \circ \pi_a = g \circ \pi_c \), we may omit this morphism. With this caveat the universal property reads:

I.e., given \( d \in b, \; \psi : d \to c, \; \varphi : d \to a \) so that

\[
\pi_c \circ \psi = \pi_b \circ \varphi = d
\]

and \( \pi_a \circ \psi = \varphi \).

Example Fiber products exist in the category \( \text{Set} \) of sets and functions.

Given \( X \times Y \in \text{Set} \), let \( X \times Y' = \{ (x,y) \in X \times Y \mid f(x) = g(y) \} \).

Define \( \pi_X : X \times Y' \to X \), \( \pi_Y : X \times Y' \to Y \) by restricting

the projections: \( \pi_X(x,y) = x \), \( \pi_Y(x,y) = y \) for all \( (x,y) \in X \times Y' \).

We check the universal property: given two functions \( \psi_X : W \to X \), \( \psi_Y : W \to Y \) with \( f \circ \psi_X = g \circ \psi_Y \) \( \exists ! \; \psi : W \to X \times Y \) with

\( \pi_X \circ \psi = \psi_X \), \( \pi_Y \circ \psi = \psi_Y \)

(by the universal property of the product \((X \times Y, \, \pi_X : X \times Y \to X, \, \pi_Y : X \times Y \to Y)\)).

Since \( f \circ \psi_X = g \circ \psi_Y \), \( \psi : W \to X \times Y \) (by the universal property of the product \((X \times Y, \, \pi_X : X \times Y \to X, \, \pi_Y : X \times Y \to Y)\)) satisfies

\( f \circ \psi_X (w) = g \circ \psi_Y (w) \), i.e. \( \psi(w) \in X \times Y \).

Remarks 1) \( axb \in \mathcal{C} \) depends on \( f \) and \( g \) even if we omit it in notation.

2) Given \( X \xrightarrow{f} Y \xleftarrow{g} Y \) in \( \text{Set} \), the fiber product \( X \times_f Y \) is \( f^{-1}(Y) \).
\[
\text{if } f^{-1}(Y) \to X \text{ is the inclusion, then } Y \text{ and given } W \to X, W \to Y \text{ so that } f \circ \phi_W = \phi_Y, \text{ then } f(\phi_W(w)) = \phi_Y(w)
\]
for all \( w \in W \). Hence \( \phi_W(w) \in f^{-1}(Y) \), which gives us \( Y = \phi_W : W \to f^{-1}(Y) \).

Note that \( f^{-1}(Y) \neq \{ (x,y) \in X \times Y \mid f(x) = y \} \) (as sets). But limits are unique only up to an isomorphism and \( h : f^{-1}(Y) \to \{ (x,y) \mid f(x) = y \} \) is a bijection.

2) If \( f : X \to Z \) is the inclusion of a subsets and \( g : Y \to Z \) is a function then
\[
X \times_{f^{-1}(Z) \uparrow \downarrow} Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \} \subseteq X \times Y
\]
being \( g \) and the inclusion, respectively:
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z
\end{array}
\]

**Example** Fiber products exist in \( \text{Top} \), the category of topological spaces and continuous maps: given \( X \xleftarrow{f} Z \xrightarrow{g} Y \) in \( \text{Top} \), we define
\[
X \times_{f^{-1}(Z) \uparrow \downarrow} Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \} \subseteq X \times Y
\]
and give it the subspace topology (the topology on \( X \times Y \) is the product topology, of course).

The space \( X \times_{f^{-1}(Z) \uparrow \downarrow} Y \) come with evident continuous maps \( \pi_X : X \times_{f^{-1}(Z) \uparrow \downarrow} Y \to X \), \( \pi_Y : X \times_{f^{-1}(Z) \uparrow \downarrow} Y \to Y \) (as in the case of sets). Check that
\[
( X \times_{f^{-1}(Z) \uparrow \downarrow} Y, \pi_X, \pi_Y )
\]
has the desired universal property.

**Definition** A category \( C \) is **complete** if for any functor \( F : I \to C \), where \( I \) is a small category, the limit of \( F \) exists.

**Theorem 15.1** Suppose a category \( C \) has all equalizers and all small products. Then \( C \) is complete.

(proof next time)

**Examples** The category \( \text{Set} \) of sets has products and equalizers.
Hence by 15.1 Set has all small limits, i.e. Set is complete.

The category Top has equalizers and small products. Hence Top is complete.

The category Vect has equalizers and small products. Hence Vect is complete.

Let's check that the category Group of groups is complete. If \( \{G_i\}_{i \in I} \) is a family of groups indexed by a set \( I \), then
\[
\prod_{i \in I} G_i = \{ x : I \to \bigcup_{i \in I} G_i \mid x(i) \in G_i \} 
\]
which is a group under "coordinate-wise" multiplication.

If \( G, K \) are two groups and \( f, h : G \to K \) are two homomorphisms consider \( L = \{ g \in G \mid f(g) = h(g) \} \).

Then \( L \neq \emptyset \) since the identity \( e \in L \). Also, given \( a, b \in L \)
\[ f(a) = h(a), \quad f(b) = h(b). \Rightarrow f(ab^{-1}) = f(a)(f(b))^{-1} = h(a)(h(b))^{-1} = h(ab^{-1}). \]
\[ \Rightarrow ab^{-1} \in L. \]

\( \therefore L \) is a subgroup of \( G \).

It's easy to check that the inclusion map \( \pi : L \to G \), \( \pi(x) = x \ \forall x \in L \) makes \( L \) into the equalizer of \( G \xrightarrow{f} K \):

For any group \( N \) and

for any homomorphism \( \varphi : N \to G \) with \( f \circ \varphi = h \circ \varphi \)
we have \( f(\varphi(n)) = h(\varphi(n)) \ \forall n \in N. \)

\( \Rightarrow \) the image of \( \varphi \) is in \( L \). Hence
\[
\exists ! \phi : N \xrightarrow{\phi} G \xrightarrow{f} H.
\]

We conclude that Group has all small limits.