Last time. Russell’s paradox: the collection of all sets is not a set.

- Freyd’s theorem: in a category $E$ if there is a product of a family of objects indexed by $E_i =$ collection of all morphisms in $E_i$ then $E$ is a preorder.
- the notion of a Grothendieck’s universe and Grothendieck’s axiom: every set in a member of a universe.
- $V$-small and $V$-locally small categories (where $V$ is a universe)

Today: back to category theory: limits and colimits.

But first: discrete categories.

**Definition** A category $E$ is discrete if the only morphisms in $E$ are identity morphisms: $\forall a, b \in E, \text{ Hom}_E(a, b) = \{ \emptyset \text{ if } a \neq b \}
\{ \text{id}_a \text{ if } a = b \}$.

Any set $C$ gives rise to a discrete category $\mathcal{C}$: the set of objects of $\mathcal{C}$ is $C$; for $a, b \in C$, $\text{Hom}_C(a, b) = \{ \emptyset \text{ if } a \neq b \}$. So $\mathcal{C} = \{ \text{id}_a \mid a \in C \}$.

**Note:** A function $f : C \to D$ between two sets gives rise to a functor $F : C \to D$ between the corresponding discrete categories. $F$ is defined by $F(a \xrightarrow{\text{id}_a} a) = f(a)$ for all $a \in C$.

This gives us a functor $i : \text{Set} \to \text{CAT}$.

In particular, given a category $E$ a collection of objects $\{ a_i \}_i$ of $E$ indexed by a set $I$, ie. a function $a : I \to E_i$, “is” a functor $a : I \to E$ where we now regard the set $I$ as a discrete category.
We are now in position to introduce limits.

**Definition** Let $F : I \to C$ be a functor from a (small) category $I$ to a category $C$. A cone on $F$ is an object $c \in C$ (the vertex of the cone) together with a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ of morphism in $C$, one for each object $i$ of $I$, so that a morphism $i \xrightarrow{f} j \in I$ the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{p_j} & F(j) \\
F(i) & \xrightarrow{F(f)} & F(j)
\end{array}
\]

commutes in $C$.

Cone on $F : I \to C$ form a category $\text{Cone}(F)$: a morphism in $\text{Cone}(F)$ from $(c, \{p_i : c \to F(i)\}_{i \in I})$ to $(c', \{p_i' : c' \to F(i)\}_{i \in I})$ in a morphism $f : c \to c'$ in $C$ so that $c \xrightarrow{f} c'$ commutes in $C$ for all $i \in I$.

**Definition** A limit of a functor $F : I \to C$ is a terminal object in $\text{Cone}(F)$. That is, it's a cone $(c, \{p_i : c \to F(i)\}_{i \in I})$ on $F$ so that for each family of morphisms $\{f_i : d \to F(i)\}_{i \in I}$ so that $f_i \circ p_i \xrightarrow{\text{commutes for all}} F(i)$

\[
\begin{array}{cc}
F(i) & \xrightarrow{F(f)} \ F(j) \\
F(i) & \xrightarrow{F(f)} \ F(j)
\end{array}
\]

exists $f : d \to c$ so that $d \xrightarrow{f} c$ commutes for all $i \in I$.

Since a limit of $F : I \to C$ is a terminal object in $\text{Cone}(F)$ (if it exists), limits are unique (up to a unique iso in $\text{Cone}(F)$).

We write $(\lim_i \{p_i : \text{lim} \to F(i)\}_{i \in I})$ or just $\text{lim}F$ to denote “the” limit of a functor $F$.

Many other notations are used for limits.

**Example** If $F : I \to C$ is a functor and $I$ is a discrete category then the limit of $F$ is the product $\prod_{i \in I} F(i)$.
"Example" let $\mathcal{C}$ be the category with two objects $x, y$ and two non-identity morphisms $x \xrightarrow{\alpha} y$.

Let $\mathcal{C}$ be a category and $F: \{x \xrightarrow{\alpha} y\} \rightarrow \mathcal{C}$ a functor. A cone on $F$ is an object $c \in \mathcal{C}$ and a pair of morphisms $p_x: c \rightarrow F(x)$, $p_y: c \rightarrow F(y)$ so that

$$
\begin{align*}
F(x) & \rightarrow F(c) \\
F(y) & \rightarrow F(c)
\end{align*}
$$

commutes, i.e.

$$
F(x) \circ p_x = p_y \quad \text{and} \quad F(y) \circ p_x = p_y.
$$

So a cone on $F$ is simply a morphism $c \xrightarrow{p} F(x)$ so that $F(x) \circ p = F(y) \circ p$ (and then $p_y = F(x) \circ p$.)

The limit of $F: \{x \xrightarrow{\alpha} y\} \rightarrow \mathcal{C}$ is an object $L \in \mathcal{C}$, a morphism $\pi: L \rightarrow F(x)$ so that

1) $F(x) \circ \pi = F(y) \circ \pi$ and

2) if $d \xrightarrow{g} F(x)$ is a cone on $F$, then $\exists! \pi: d \rightarrow L$ in $\mathcal{C}$ so that $d \xrightarrow{g} F(x)$ commutes.

$$
\begin{align*}
\begin{array}{c} d \\ \downarrow \pi \\ L \end{array} & \xrightarrow{1} F(x) \\
\end{align*}
$$

(And then $d \xrightarrow{g} F(x)$ commutes as well.)

The limit of $F: \{x \xrightarrow{\alpha} y\} \rightarrow \mathcal{C}$ is called the equalizer of $F(x) \xrightarrow{F} F(y)$.

Note. Given a category $\mathcal{C}$, two objects $a, b \in \mathcal{C}$ and two morphisms $a \xrightarrow{p} b$ there is unique $F: \{x \xrightarrow{\alpha} y\} \rightarrow \mathcal{C}$ with $F(x) = a$, $F(y) = b$, $F(x) = f$, $F(y) = g$.

So given $a \xrightarrow{\alpha} b$ in $\mathcal{C}$ the limit of $F$ is often called the equalizer of $a \xrightarrow{\alpha} b$ (with the functor $F$ suppressed). $a \xrightarrow{\alpha} b$ is called a diagram in $\mathcal{C}$ (of "shape" $x \xrightarrow{\alpha} y$).

Claim. For any two sets $X, Y$, and any two functions $f, g: X \rightarrow Y$ the equalizer of $X \xrightarrow{f} Y$ exists.
Proof Let $L = \{ x \in X \mid f(x) = g(x) \}$ and let $p: L \to X$ be the inclusion map: $p(x) = x$.

If $D$ is any set and $g: D \to X$ a function with $f \circ g = g \circ f$ then $f(g(d)) = g(g(d))$

$\forall d \in D \Rightarrow g(d) \in L. \Rightarrow \exists! m: D \to L$ which is given by $m(d) = g(d) \ \forall d \in D$

$\Rightarrow \ L \overset{p}{\hookrightarrow} X$ is the equalizer of $X \overset{f}{\to} Y$.

Claim Let $\textbf{Vec}$, the category of vector spaces and $V \xrightarrow{T} U$ a diagram in $\textbf{Vec}$ (i.e. a pair of vector spaces and a pair of linear maps).

Then the equalizer $V \overset{T}{\underset{S}{\longrightarrow}} U$ exists.

Proof Let $L = \{ v \in V \mid T(v) = S(v) \}$. Then $L$ is a subspace of the vector space $V$ (check it!) and the inclusion $p: L \to V$, $p(v) = v \ \forall v \in L$ is linear.

If $R: W \to V$ is a linear map with $T \circ R = S \circ R$ then $T(R(v)) = S(R(v))$

for all $w \in W$. So $R$ is really a map $R: W \to L$.

\[ \Rightarrow \ p: L \to V \text{ is "the" equalizer of } V \overset{T}{\underset{S}{\longrightarrow}} U. \]

Claim For any diagram $X \overset{f}{\longrightarrow} Y$ in $\textbf{Top}$ the equalizer exists. Proof

Let $L = \{ x \in X \mid f(x) = g(x) \}$. It's a subset of $X$. Give $L$ the subspace topology.

$U \subseteq L$ is open $\Rightarrow \tilde{U} \subseteq X$ open with $\tilde{U} \cap L = U$. Then the inclusion map $p: L \to X$ in continuous:

$\forall U \subseteq X$ open, $p^{-1}(U) = \tilde{U} \cap L$.

If $\tilde{U}$ is another topological space and $h: \tilde{Z} \to \tilde{X}$ a continuous map with $f \circ h = g \circ h$ then $h(\tilde{Z}) \subseteq \tilde{L}$, so we can regard $h$ as a function from $\tilde{Z}$ to $L$. We need to check continuity.

Let $\tilde{U} \subseteq \tilde{L}$ be open. By definition of the subspace topology, $\tilde{U} = \tilde{L} \cap \tilde{U}$ for some open set $\tilde{U} \subseteq \tilde{X}$. Since $h: \tilde{Z} \to \tilde{X}$ is continuous, $h^{-1}(\tilde{U})$ is open in $\tilde{Z}$. But $h^{-1}(\tilde{U}) = \tilde{h}(U)$. $\Rightarrow$ $h: \tilde{Z} \to L$ is continuous and so $L \overset{p}{\hookrightarrow} X$ is the equalizer of $X \overset{f}{\longrightarrow} Y$.

Definition Let $\mathcal{C}$ be a category and $F: \mathcal{C} \xrightarrow{\alpha} \mathcal{D}$ a functor. The limit of $F$ is called the fiber product of the diagram

\[ \begin{array}{ccc}
F(\alpha) & \xrightarrow{\beta} & F(\beta) \\
\downarrow & & \downarrow \\
F(\alpha) & \to & F(\beta)
\end{array} \]

in $\mathcal{C}$. 


Equivalently, let $E$ be a category, $a, b, c \in E$ three objects and $f, g : a \to b$ two morphisms in $E$. The fiber product of $a \xrightarrow{f} b$ in $E$ is an object $a_b c = a_{x, b, c}$ of $E$ and a pair of morphisms $p_a : a_b c \to a$, $p_b : a_b c \to b$ so that 1) $a \xrightarrow{f} b \xrightarrow{g} c$ commutes, and 2) given a commuting diagram $d \xrightarrow{f} a \xrightarrow{g} b$ there exists a unique morphism $d \xrightarrow{c} a_b c$ so that $d \xrightarrow{c} a_b c \xrightarrow{p_a} a \xrightarrow{f} b$ commutes.

Example. Fiber products exist in $\text{Vect}$: given a pair of linear maps $T : V \to U$, $S : W \to U$ define $V_x W = \{(v, w) \in V \times W \mid T(v) = S(w)\}$. It's not hard to check that $V_x W$ is a subspace of $V \times W$.

Moreover, given a pair of linear maps $K : Z \to V$, $H : Z \to W$ so that $T K = S H$, it's easy to see that $V_x U W$, $(K(z), H(z)) \in V_x W$. So we get a (unique) linear map $\varphi : Z \to V_x W$ given by $z \mapsto (K(z), H(z))$. \qed