Last time: examples of products -
- In a poset, $\prod_{i \in I} x_i = \text{inf}_{i \in I} x_i = \text{greatest lower bound of } x_i$.

In Vect, a product exists $= \prod_{i \in I} \mathbb{R}^i = \{ (x_i)_{i \in I} : x_i \in \mathbb{R}^i \}$. We need a notion of a subbasis:

Let $X$ be a set, a subset $A \subseteq \mathcal{P}(X)$ is a subbasis if $\bigcup A = X$. Then $\mathcal{B} = \{ \sum_{i=1}^k S_i, \cap_{i=1}^k S_i : k \geq 1, S_i \in A \}$ is a basis for a topology on $X$.

We used it to construct the product topology on $\prod_{i \in I} X_i$ where $\mathcal{G}(X_i, T_i)$ is a family of topological spaces.

Finally, we defined a subcategory $\mathcal{B}$ of a category $\mathcal{C}$ and observed that images of functors need not be subcategories.

Universal property of products in terms of Hom's:
Recall $\prod_{i \in I} f_i : \prod_{i \in I} X_i \to X_{i_0}$ is a product in $\mathcal{C} \iff$
\[(*) \quad \forall f : d \to \prod_{i \in I} X_i \exists ! f_i : d \to X_i \text{ st. } p_i \circ f = f_i \quad \forall i.

This is equivalent to:
\[\forall d \in \mathcal{C}
\[
\begin{align*}
\Phi : \text{Hom}_\mathcal{C}(d, \prod_{i \in I} X_i) & \to \prod_{i \in I} \text{Hom}_\mathcal{C}(d, X_i), \\
f & \mapsto (f_i).
\end{align*}
\]

is a bijection.
(Here I'm pretending that $\forall a, b \in \mathcal{C}$, Hom$_\mathcal{C}(a, b)$ is a set; we'll see later on that this pretense is OK to do.)

**Coproducts**

**Definition.** Let $\mathcal{C}$ be a category, $\mathfrak{a} : I \to \mathcal{C}$ a family of objects in $\mathcal{C}$ indexed by a set $I$. A coproduct of $\mathfrak{a}$ (if it exists) is the product of $\mathfrak{a}$, $\coprod_{i \in I} \mathfrak{a}_i \in \mathcal{C}$. That is, it's an object $C$ of $\mathcal{C}$ together with a family of morphisms
\[\mathfrak{a}_i : \mathfrak{a}_i \to C \quad \text{so that given any object } d \text{ of } \mathcal{C}\]
and any family of morphisms \( f_j : a_j \to d_{j+1} \) \( j \in \mathbb{J} \) \( f : c \to d \) 

So that \( f_{ij} = f_j \circ f_i \) \( \forall i, j \in \mathbb{J} \) \( i \leq j \) \( C \xrightarrow{f_i} C \xrightarrow{f_j} D \).

Coproducts (when they exist) are unique up to a unique isomorphism. We write \( \bigsqcup a_i \) \( (i : a_j \to \bigsqcup a_i \downarrow f_{ij}) \) or just \( \bigsqcup a_i \) for the coproduct of \( a_i \downarrow i \in \mathbb{J} \).

Ex. Coproducts exist in Set; they are disjoint unions.

For example, given a collection \( \{ a_i \}_{i \in \mathbb{J}} \) of sets we can define \( \bigsqcup a_i = \bigcup_{i \in \mathbb{J}} a_i \times \{ i \} \) and \( \pi_j : a_j \to \bigsqcup a_i \) by \( \pi_j(x) = (x, j) \) \( \forall x \in a_j \).

Given a collection of functions \( (f_j : a_j \to d_{j+1})_{j \in \mathbb{J}} \) we define \( f : \bigsqcup a_i \to d \) by \( f(x, i) = f_j(x) \) \( \forall j \in \mathbb{J}, \ x \in a_j \).

Ex. Coproducts exist in Vect, the category of vector spaces. They are direct sums and are usually constructed as follows: given a family \( \{ V_i \}_{i \in \mathbb{I}} \) of vector spaces set \( \bigoplus V_i = \{ \xi \}_{i \in \mathbb{I}} \in \prod_{i \in \mathbb{I}} V_i \) \( \xi_i = 0 \) for all but finitely many \( i \in \mathbb{I} \).

The maps \( \pi_j : V_j \to \prod_{i \in \mathbb{I}} V_i \) are given by \( \pi_j(x) = (x_i)_{i \in \mathbb{I}} \) where \( x_i = 0 \) \( \forall i \in \mathbb{I}^j \).

Given a family of linear maps \( \{ T_i : V_i \to W \}_{i \in \mathbb{I}} \) the corresponding map \( T : \bigoplus V_i \to W \) should satisfy \( T \circ \pi_j = T_j \). So we define: for \( \xi \in \bigoplus V_i \) \( T((\xi_i)_{i \in \mathbb{I}}) = \sum_{i \in \mathbb{I}} T_i(\xi_i) \).

Note: Since \( (\xi_i)_{i \in \mathbb{I}} \in \bigoplus V_i \), \( f \in \mathbb{J} \) and \( j = j + 1 \).
So that \( x_i = 0 \) for \( i \neq j \). 

\[ T \left( \langle x_i \rangle_{i \in I} \right) = T_{j_1} (x_{j_1}) + \cdots + T_{j_k} (x_{j_k}) , \] which is finite and so makes sense.

**Remark.** The free vector space \( F(X) \) on a set \( X \) is the direct sum \( \bigoplus \mathbb{R} \) (one \( \mathbb{R} \) for each \( x \in X \)).

**Remark.** If \( I = \emptyset \), \( \mathbb{N}_0 \) is a terminal object in \( \mathcal{G}^\text{op} \), hence an initial object in \( \mathcal{G} \).

**Example.** If \( (P, \leq) \) is a poset and \( \{ x_i \} \) is a family of elements of \( P \), then \( \bigwedge \{ x_i \} \) is the least upper bound of \( \{ x_i \} \) in \( P \).

(ii) if it exists (of course).

**Example.** Coproducts exist in \( \text{Top} \) : given a family \( \{ (X_i, T_i) \}_{i \in I} \) of topological spaces their coproduct in \( (\bigsqcup_{i \in I} X_i, T) \) where \( T \) is defined by

\[ U \subseteq T \iff \bigcup_{i \in I} X_i = T_i \quad \forall i \in I. \]

(Here I pretend that \( X_i \in \bigsqcup_{i \in I} X_i \).)

Note that \( T \) is generated by the basis \( B = \bigsqcup_{i \in I} T_i \).

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Size (sometimes) matters.

So far we have been ignoring theories of sets/collections. But sometimes size of collections matters and we'd need to be more careful.

First of all, what does one mean by "size"?

**Definition.** Two collections \( X \) and \( Y \) have the same size (we write \( |X| = |Y| \)) if there is an invertible map \( f: X \to Y \).
The size of $X$ is less than or equal to the size of $Y$ (we write $|X| \leq |Y|$) if there is an injective map $f : X \to Y$.

Schröder-Bernstein theorem guarantees that if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$; this is not obvious.

We say that $Y$ is strictly bigger than $X$ (and write $|X| < |Y|$ or $|X| \subsetneq |Y|$) if $f$ an injection $X \to Y$ and no bijection $X \to Y$.

**Theorem (Cantor) For any set / collection $X$, $|X| \leq |\mathcal{P}(X)|$:** $X$ is strictly smaller than the collection $\mathcal{P}(X)$ of its subsets.

**Proof.** The function $f : X \to \mathcal{P}(X), f(x) = \{x\}$ is injective (since $1 \times 1 = \{y\} \iff x = y$).

Suppose there is an invertible function $f : X \to \mathcal{P}(X)$. Then $f$ is surjective. Consider

$Y = \{ x \in X | x \notin f(x) \}.$

Since $f$ is surjective, $Y = f(x_0)$ for some $x_0 \in X$.

If $x_0 \in Y$ then $x_0 \notin f(x_0) = Y$. Contradiction.

If $x_0 \notin Y = f(x_0)$, then $x_0 \in Y$. Contradiction again. D

**Next time:** Russell's paradox — the collection $V$ of all sets is not a set.

**Note.** For any set $X$, the function $f : X \to V, f(x) = \{x\}$ is injective so $|X| \leq |V|$. For any set $X$.

It follows from Russell's that given a set $X$ there is no bijection $X \to V$. Hence $V$ is bigger than any set.