Last time: *Notions of continuity: from ε-δ definition of continuity
to "\( f : (x, T_x) \to (y, T_y) \) is continuous if \( U \subseteq Y \) open, \( f^{-1}(U) \) is open"*

- Subspace topology: If \( (X, T_X) \) is a top space, \( Y \subseteq X \),
  - smallest topology \( T_Y \) on \( Y \) so that \( i : (Y, T_Y) \to (X, T_X) \), \( i(Y) = Y \)
in continuous: \( U \subseteq Y \) open \( \iff \exists U \subseteq X \) open with \( \tilde{U} \cap Y = U \).
- A subset \( B \) of a topology \( T \) on a set \( X \) is a basis
  of \( T \) if \( \forall U \in T \) is a union of elements of \( B \).

Now suppose we have a set \( X \) and \( B \subseteq P(X) \), is \( B \) a basis
for some topology \( T \) on \( X \)?

**Lemma 10.1** Let \( X \) be a set, \( B \subseteq P(X) \), \( B \) is a basis for a topology
\( T \) on \( X \) if

1. \( \bigcup B = X \) and \( \forall B_1, B_2 \in B \)
   \( B_1 \cap B_2 \) is a union of elements of \( B \).

**Proof** "Bearch" \( \forall A \subseteq P(X), \bigcup A = \{ x \in X \mid x \in A \text{ for some } A \in A \} \)

If \( B \) is a basis for a topology \( T \), then

\( \forall U \subseteq X \exists A \subseteq B \) with \( U = \bigcup A \).

So define \( T \subseteq P(X) \) by (1). Let's check that \( T \) is a topology

1. \( \emptyset \subseteq T \) since \( U \emptyset = \emptyset \) and \( \emptyset \subseteq B \).
2. \( X \subseteq T \) since \( UB = X \)
3. Suppose \( U, V \subseteq T \). Then \( \exists A, B \subseteq B \) so that

   \( U = \bigcup_{B \in A} U_B = UA, \quad V = \bigcup_{B' \in B} U_{B'} = UB' \)

   \( \bigcup V = (\bigcup B) \cap (\bigcup B') = U(B \cap B') \).

But \( U \cup V \subseteq UB \cap UB' \subseteq UB \cap UB' \subseteq UB \cap UB' \) for some \( \exists D \subseteq B \).

Similarly if \( \bigcup_{D \subseteq B} \subseteq T \) then \( \exists A, B \subseteq B \) so that \( (\bigcup A) \subseteq U \).
Remark 10.2 Let \((X, \mathcal{T}_x), (Y, \mathcal{T}_y)\) be two topological spaces and \(B \subseteq \mathcal{T}_y\) is a basis for \(\mathcal{T}_y\). Then \(f : (X, \mathcal{T}_x) \to (Y, \mathcal{T}_y)\) is continuous if and only if \(\forall B \in B, f^{-1}(B)\) is open.

Proof (\(\Rightarrow\)) Since \(B \subseteq \mathcal{T}_y\) and \(f\) is continuous, \(\forall B \in B, f^{-1}(B)\) is open.

(\(\Leftarrow\)) Conversely, if \(U \in \mathcal{T}_y\), \(\exists A \in B\) so that \(U = \bigcup A\).

Hence

\[
f^{-1}(U) = \bigcup_{A \in B} f^{-1}(B).
\]

Products and coproducts in \(\text{Top}\).

Let \((X, \mathcal{T}_x), (Y, \mathcal{T}_y)\) be two topological spaces. Their product \((X \times Y, \mathcal{T}_{X \times Y})\) is a topological space together with two continuous maps \(p_x : (Z, \mathcal{T}_z) \to (X, \mathcal{T}_x), p_y : (Z, \mathcal{T}_z) \to (Y, \mathcal{T}_y)\). So that \((Z, \mathcal{T}_{X \times Y})\) and any two continuous maps \(f : W \to X, g : W \to Y\) are continuous \(p_x \circ f = f, p_y \circ g = g\).

Construction Let \(Z = X \times Y\). \(p_x : X \times Y \to X\) the projection \((x, y) \mapsto x\), \(p_y : X \times Y \to Y\) the projection \((x, y) \mapsto y\). We need \(p_x, p_y\) be continuous. So if \(U \subseteq X\) is open, \((p_x)^{-1}[U] = U \times Y\) should be open in the topology \(\mathcal{T}_{X \times Y}\). Similarly \(X \times V\) should be in \(\mathcal{T} = \bigcup_{V \in \mathcal{T}_y} V \times \mathcal{T}_y\).

Claim \((U \times Y) \cup (X \times V) = U \times V\) should be in \(\mathcal{T}_{X \times Y}\).

Check By 10.1 we need to check \(U \in \mathcal{T}_x, V \in \mathcal{T}_y\) is a basis for a topology \(\mathcal{T}_{X \times Y}\) on \(X \times Y\). Check 21 if \(B_1, B_2 \in B\), \(B_1 \cup B_2\) is a union of elements of \(B\).
Now, since \( x, y \in X, \) \( y \in Y, \) \( x \times y \in B. \) \( \Rightarrow U_B = X \times Y. \)

\[ U \text{ is a basis for a topology } T \text{ on } X \times Y. \]

\( T \) is called the product topology. (on \( X \times Y \))

**Note**

\( U \in X \text{ open, } p_X(U) \in B \) \( \text{CT. } \Rightarrow p_X : X \times Y \to X \)

\( \text{is continuous. Similarly } p_Y : X \times Y \to Y \text{ is continuous.} \)

**Universal property?**

If \( f : W \to X, \) \( g : W \to Y \) are continuous, \( \exists ! \) \( (f, g) : W \to X \times Y \)

So that \( p_X \circ (f, g) = f, \) \( p_Y : (f, g) = g \)

\[ (f, g) : (f(w), g(w)) \]

Is \( (f, g) \) continuous? Given \( U \in T_X, \) \( V \in T_Y \)

\[ (f, g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \]

\( f^{-1}(U), g^{-1}(V) \) are open in \( W. \) \( \Rightarrow (f, g)^{-1}(U \times V) = f^{-1}(U) \cup g^{-1}(V) \)

is open in \( W. \) By Remark 10.2, \( (f, g) \) is continuous.

We conclude that \( X \times Y \) with the product topology and

the projections \( p_X, p_Y \) is the product of \( (X, T_X) \) and \( (Y, T_Y) \)

in \( \textbf{Top}. \)

**coproducts**

Given \( (X, T_X), (Y, T_Y) \) their coproduct \( (Z, T_Z) \)

exists and is constructed as follows:

as a set \( Z = X \cup Y, \)

\( \beta = \{ U \cup V | U \in T_X, V \in T_Y \} \)

is a basis for a topology \( T_Z \) on \( Z = X \cup Y. \)

Moreover, the canonical inclusion \( i_X : X \to X \cup Y, \)

\( i_Y : Y \to X \cup Y \)

are continuous. For example \( (i_X^*)(U \cup V) = U \) for all \( U \cup V \in \beta. \)

Finally, if \( f : X \to W, \) \( g : Y \to W \) are continuous

and \( \emptyset \in T_W \) is open then

\[ (f \cup g)^{-1}(\emptyset) = f^{-1}(\emptyset) \cup g^{-1}(\emptyset), \] which is open

in \( (X \cup Y, T_Z). \)

\[ f \cup g : X \cup Y \to W \text{ is continuous.} \]
Products in general

Definition. Let \( C \) be a category, \( I \) a set and \( \{a_i : i \in I\} \) a collection of objects of \( C \). The product of the family \( \{a_i : i \in I\} \) (if it exists) in an object \( C \) of \( C \) together with a family of morphisms
\[ \pi_i : C \to a_i, \quad i \in I \]
with the following universal property:

for any object \( d \in C \) and any family \( \{f_i : d \to a_i : i \in I\} \) of morphisms, \( \exists \) unique morphism \( \varphi : d \to C \) so that
\[ \pi_i \circ \varphi = f_i \quad \forall i \in I. \]

Remarks:
(i) If \( I = \{1, 2\} \), then the product of \( \{a_1, a_2\} \) = \( a_1 \times a_2 \) in the binary product \( a_1 \times a_2, \pi_1(a_1, a_2) = a_1, \pi_2(a_1, a_2) = a_2 \).

(ii) \( I \) a product \( (c, \{\pi_i : c \to a_i : i \in I\}) \) exists, it's unique up to a unique iso. The proof is the same as in the case of binary products.

We therefore denote it by \( \Pi a_i \) \( i \in I \) and refer to it as "the" product of the family \( \{a_i : i \in I\} \).

(iii) If \( I = \emptyset \), then \( \Pi a_i \) is an object \( C \in C \) so that \( \forall d \in C \)
\[ \exists \varphi : d \to C \]. In other words, a product of an empty family of objects of \( C \) is a terminal object of \( C \).

(iv) \( I = \{1\} \), a 1-element set then \( \Pi a_1 = a_1 \) and \( \pi_1 : a_1 \to a_1 \)
the identity map.

Exercise. For any set \( I \) and any family \( \{a_i : i \in I\} \) in Set
\[ \Pi a_i \] exists (provided we accept the axiom of choice).

One defines \( \Pi a_i = \{x : I \to \bigvee_{i \in I} a_i \mid x(i) \in a_i, \forall i \in I\} \)
\[ \pi_i : \Pi a_i \to a_i \] is \( \pi_i(x) = x(i) \quad \forall i \in I. \) Universal property is easy.
Given a set $d$ and a family of functions $(f_i : d \to a_i)_{i \in I}$

Define $f : d \to \prod_{i \in I} a_i$ so that $(P_j \circ f)(y) = f_j(y)$ for $j \in I$.

Namely, define $f(y) = \prod_{i \in I} a_i = \{ x : I \to \cup_{i \in I} a_i \}$ by

$$(f(y))(j) = f_j(y)$$