1. Let \( f : G \rightarrow H \) be a homomorphism between two groups and let \( L \) be a subgroup of \( H \). Prove that the preimage

\[
f^{-1}(L) := \{ g \in G \mid f(g) \in L \}
\]
is a subgroup of the group \( G \).

2. Is the subset \( H \) of the group \( GL(2, \mathbb{R}) \) consisting of two by two invertible matrices with integer entries a subgroup of \( GL(2, \mathbb{R}) \)? Explain/prove your answer.

3. Consider the matrices \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). They are both elements of the group \( GL(2, \mathbb{R}) \). What are the subgroups \( \langle A \rangle \) and \( \langle B \rangle \) that they generate? In particular what sets do these subgroups consist of?

4. (a) Prove that the set

\[
T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{R}, \; ad \neq 0 \right\}
\]
is a subgroup of \( GL(2, \mathbb{R}) \).

(b) Is the group \( T \) abelian (i.e., commutative)? Explain.

5. Let \( G \) be a group, \( a, b \in G \) two elements. Prove that

\[
(ab)^{-1} = b^{-1}a^{-1}.
\]

Hint: inverses are unique, and \((ab)b^{-1}a^{-1} = \ldots\).

6. Let \( G \) be a collection of function from \( \mathbb{R} \) to \( \mathbb{R} \) of the form

\[
f(x) = ax + b
\]
for some \( a, b \in \mathbb{R} \) with \( a \neq 0 \), the set of affine functions.

(a) Prove that \( G \) is a group under the composition of functions.

(b) Prove that the map \( \varphi : G \rightarrow GL(2, \mathbb{R}) \) defined by

\[
\varphi(ax + b) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}
\]
is an injective homomorphism.

7. Prove that the composite \( g \circ f : G \rightarrow L \) of two homomorphisms \( f : G \rightarrow H \) and \( g : H \rightarrow L \) is again a homomorphism. Prove that the identity map \( id_G : G \rightarrow G \) is a homomorphism.

8. Let \( G \times X \rightarrow X \) be an action of a group \( G \) on a set \( X \).
(a) Prove that for any $g \in G$ the map $\varphi_g : X \to X$ defined by $\varphi_g(x) := g \cdot x$ is invertible. Hint: show that $\varphi_{g^{-1}}$ is an inverse of $\varphi_g$.

(b) Prove that the map

$$\varphi : G \to \text{Sym}(X), \quad g \mapsto \varphi_g$$

is a homomorphism. Here as in the lectures $\text{Sym}(X)$ is the group of bijections of the set $X$. 