Homework #10, Math 427, Prof. Eugene Lerman
Due Wednesday, November 6, 2019 in class

Unless noted otherwise all the rings have 1, all subrings contain 1, and all ring homomorphisms preserve 1.

1 Suppose $R$ is a ring with unity $1_R$. Let $S$ be the subring generated by $1_R$. That is, $S$ is the intersection of all the subrings of $R$ (by our convention all of them contain $1_R$). Show that either $S$ is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_n$ for some $n$.

Show that if $R$ is an integral domain and the subring generated by $1_R$ is isomorphic to $\mathbb{Z}_n$ then $n$ has to be prime.

2 Recall that the product ring $R \times S$ of two rings $R$ and $S$ is the abelian group $R \times S$ with the multiplication defined coordinate-wise:

$$(a,b)(a',b') = (aa',bb').$$

Suppose $I \subset R$ and $J \subset S$ are ideals. Show that $I \times J$ is an ideal in the product ring $R \times S$. Prove that $(R \times S)/(I \times J)$ is isomorphic to $(R/I) \times (S/J)$.

3 Let $I$ be an ideal in a ring $R$. Show that the quotient ring $R/I$ is commutative if and only if $ab - ba \in I$ for all $a,b \in R$.

4 Consider the ring $R = \mathbb{Z}[i]$ of Gaussian integers. It is defined by

$$\mathbb{Z}[i] := \{a + ib \in \mathbb{C} \mid a,b \in \mathbb{Z}\}.$$ 

Let $I$ be the ideal generated by $2 + i$: $I = (2 + i)\mathbb{Z}[i]$. Prove that $R/I$ is isomorphic to $\mathbb{Z}/5$. Hint: $i + I = (-2) + I$ in $R/I$. Use this to prove that the map $\varphi : \mathbb{Z} \to R/I$ given by $\varphi(n) = n + I$ is onto. Now compute its kernel.

5 Let $R$ be a ring with 1 and $S \subset R$ is a subring (recall that $S$ contains $1_R$). Show that any unit of $S$ is a unit of $R$. Give an example to show that the converse is not true: there is a ring $R$ with a subring $S$, and an element $u \in S$ which is not a unit in $S$ but is a unit in $R$.

6 Let $R$ be a commutative ring. An element $x \in R$ is nilpotent if $x^k = 0$ for some natural number $k$. For example 2 is nilpotent in $\mathbb{Z}_8$.

(a) Show that the set $N$ of nilpotent elements of $R$ is an ideal in $R$.

(b) Show that $R/N$ has no nonzero nilpotent elements.

(c) Show that if $S$ is an integral domain and $\varphi : R \to S$ is a homomorphism then the set $N$ of nilpotent elements is contained in the kernel of $\varphi$.

7 Write the ring $\mathbb{Z}_3$ as $\{0, 1, 2\}$. Write down the multiplication table for the quotient ring $\mathbb{Z}_3[x]/(x^2 + x + 2)$.

Hints: it is convenient to write $a$ instead of $a + (x^2 + x + 2)$ for any $a \in \mathbb{Z}_3$. It
may also convenient to set $\alpha := x + \langle x^2 + x + 2 \rangle$. Observe that $\alpha^2 + \alpha + 2 = 0$ in $\mathbb{Z}_3[x]/(x^2 + x + 2)$.

8 Let $R$ be an integral domain. Prove that the ring of polynomials $R[x]$ with coefficients in $R$ is also an integral domain.