Let $f : G \to H$ be a homomorphism between two groups and let $K$ be a subgroup of $G$. Prove that the image

$$f(K) := \{ f(g) \mid g \in K \}$$

is a subgroup of the group $H$.

Let $f : G \to H$ be a homomorphism between two groups and let $L$ be a subgroup of $H$. Prove that the preimage

$$f^{-1}(L) := \{ g \in G \mid f(g) \in L \}$$

is a subgroup of the group $G$.

Is the subset $H$ of the group $GL(2, \mathbb{R})$ consisting of two by two invertible matrices with integer entries a subgroup of $GL(2, \mathbb{R})$? Explain/prove your answer.

Consider the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

They are both elements of the group $GL(2, \mathbb{R})$. What are their orders? Explain/prove your answers.

(a) Prove that the set

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{R}, \quad ad \neq 0 \right\}$$

is a subgroup of $GL(2, \mathbb{R})$.

(b) Is the group $T$ commutative? Explain.

Let $G$ be a collection of function from $\mathbb{R}$ to $\mathbb{R}$ of the form

$$f(x) = ax + b$$

for some $a, b \in \mathbb{R}$ with $a \neq 0$, the set of affine functions.

(a) Prove that $G$ is a group under the composition of functions.

(b) Prove that the map $\varphi : G \to GL(2, \mathbb{R})$ defined by

$$\varphi(ax + b) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

is an injective homomorphism.