Let \( U \subseteq \mathbb{R}^n \) be open. A \( k \)-form \( \eta \in \Omega^k(U) \) is closed if \( d\eta = 0 \); it is exact if \( \eta = df \) for some \( f \in \Omega^{k-1}(U) \). By convention \( f \in \Omega^0(U) \) is exact \( \iff f = 0 \).

#1 Prove that exact forms are closed.

#2 Prove that if \( U \) is connected and \( f \in \Omega^0(U) \) is closed then \( f \) is constant.

#3. Prove that \( \alpha = \frac{y}{x^2+y^2} \, dx - \frac{x}{x^2+y^2} \, dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\}) \) is closed but not exact. Hint: (a) Prove \( \int_{S^1} \alpha \neq 0 \) where \( S^1 \equiv \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \} \) with any parameterization (say \( y(\theta) = (\cos \theta, \sin \theta), 0 \leq \theta \leq 2\pi \)). (b) Prove that for any \( f \in \Omega^0(\mathbb{R}^2 \setminus \{(0,0)\}) \), \( \int_{S^1} df = 0 \).

#4 Prove that if \( \alpha, \beta \) are closed, then so is \( \alpha \wedge \beta \).

#5 Prove that if \( \alpha \) is closed and \( \beta \) is exact then \( \alpha \wedge \beta \) is exact. Hint: Suppose \( \beta = df \). Then \( d(\alpha \wedge \beta) = \ldots \).

#6 Let \( F: V \to U \) be a \( C^\infty \) map \( (V \subseteq \mathbb{R}^k \) open). Prove that if \( \alpha \in \Omega^k(U) \) is closed, then \( F^*\alpha \) is closed; and if \( \alpha \) is exact then \( F^*\alpha \) is exact.