I expect you to be able to write down definitions: e.g. least upper bound, subsequence, limit of a sequence, metric, topology, lim sup, interior, closure... (this is not an exhaustive list).

1. Let $S \subset \mathbb{R}$ be a bounded set. Prove that there is a sequence $\{a_n\} \subset S$ that converges to $x = \inf S$.

2. Does the sequence $a_n = \sqrt{n^2 + 2} - n$ converge? Prove your answer. You may use limit theorems.

3. Suppose $\{x_n\}, \{y_n\}$ are two convergent sequence of real numbers with $x_n \leq y_n$ for all $n$. Prove that $\lim x_n \leq \lim y_n$.

4. Suppose $\{a_n\} \subset \mathbb{R}$ is a sequence bounded above. Is there $c \in \mathbb{R}$ with $c = \limsup a_n$? Justify your answer.

5. Consider $\mathbb{R}$ with the discrete metric $d$: $d(x, y) = 1$ for all $x \neq y$.
   a. Is the metric space $(\mathbb{R}, d)$ complete? Explain (i.e., prove your answer).
   b. What are the compact subsets of $(\mathbb{R}, d)$? Explain.

6. Let $(S, d)$ be a metric space and $\{a_n\} \subset S$ is a convergent sequence. Prove that $\{a_n\}$ is Cauchy.

7. Let $(S, d)$ be a metric space and $\{a_n\} \subset S$ is a Cauchy sequence. Prove that $\{a_n\}$ is bounded.

8. Suppose $\{x_n\}, \{y_n\}, \{z_n\}$ are three sequence of real numbers with $x_n \leq y_n \leq z_n$ for all $n$. Suppose $\{x_n\}$ and $\{z_n\}$ are convergent. Is $\{y_n\}$ necessarily convergent? Explain.

9. Let $(S, d)$ be a metric space, $C \subset S$ a closed subset and $\{a_n\} \subset S$ a sequence that converges to $L \notin C$. Prove that there is $N$ so that $a_n \notin C$ for $n > N$.

10. Consider $\mathbb{R}$ with the standard topology. Prove that the closure of $\mathbb{Q}$ is all of $\mathbb{R}$ and that $\mathbb{Q}$ has empty interior.

11. Prove that limits of sequences in metric spaces are unique.

12. Let $(S, d)$ be a metric space, $\{a_n\} \subset S$ a convergent sequence and $\{a_{n_k}\}$ a subsequence. Prove that $\{a_{n_k}\}$ converges and that $\lim a_{n_k} = \lim a_n$.

13. Let $a_n = \cos\left(\frac{2n\pi}{3}\right)$. What are $\limsup a_n$? $\liminf a_n$? Here is a more challenging problem: what if $a_n = \cos(n)$? What are $\limsup a_n$ and $\liminf a_n$?

14. Let $(S_1, d_1), (S_2, d_2)$ be two metric spaces. Is
   
   $$d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(x_2, y_2)$$
   
   for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$

   a metric on the product $S_1 \times S_2$? Prove your answer.