Last time: monotone sequences

- if \( \{S_n\} \) is increasing and bounded above, \( \{S_n\} \) converges
- if \( \{S_n\} \) is increasing and not bounded above, \( \lim S_n = +\infty \)

Similarly, if \( \{S_n\} \) is decreasing then
- \( \{S_n\} \) converges if \( \{S_n\} \) bounded below
- diverges to \(-\infty\) if \( \{S_n\} \) is not bounded below

- \( \limsup, \liminf \)

Big idea: if \( \{S_n\} \) is a sequence then
\[
V_n = \sup \{ S_k \mid k \geq n \} \quad \text{are monotone (if exist)}
\]
\[
U_n = \inf \{ S_k \mid k \geq n \}
\]

So we get
\[
\limsup S_n \leq \lim V_n \leq \liminf S_n \leq \lim U_n.
\]

These always exist (or are \(+\infty, -\infty\) resp.)

Remark
\[
\inf \{ S_k \mid k \geq n \} \leq S_{n+1} \leq \sup \{ S_k \mid k \geq n \}
\]
for all \( n \).

Theorem 10.7 Let \( \{S_n\} \in \mathbb{R} \) be a sequence

(i) If \( \{S_n\} \) converges (or diverges to \(+\infty\), or diverges to \(-\infty\))
then \( \lim \inf S_n = \lim S_n = \lim \sup S_n \)

(ii) If \( \lim \inf S_n = \lim \sup S_n \) then \( \lim S_n \) exists
(as a number or \(+\infty\) and
\[
\lim \inf S_n = \lim S_n = \lim \sup S_n
\]
Proof. We consider the finite case only: \( \limsup S_n, \liminf S_n \neq \pm \infty \).

(i) Suppose \( \lim S_n = L \). Then \( \forall \varepsilon > 0 \) \( \exists N \) so that
\[
\forall n > N, \quad |S_n - L| < \frac{\varepsilon}{2}
\]
Let \( V_M = \sup \{ S_n | n > M \} \).

Claim: \( \forall M \geq N, \quad |V_M - L| < \varepsilon \)

Proof of claim: for \( n > N \)
\[
|S_n - L| < \frac{\varepsilon}{2}
\]
\[
\Rightarrow L - \frac{\varepsilon}{2} < S_n < L + \frac{\varepsilon}{2}
\]
\[
\Rightarrow L - \frac{\varepsilon}{2} \text{ is not an upper bound of } \{ S_n | n > M \} \quad \forall M \geq N
\]
\[
L + \frac{\varepsilon}{2} \text{ is an upper bound of } \{ S_n | n > M(\forall M \geq N)
\]
\[
\Rightarrow (L - \frac{\varepsilon}{2})L - \frac{\varepsilon}{2} < \sup \{ S_n | n > M \} \leq L + \frac{\varepsilon}{2} < L + \varepsilon
\]
\[
\Rightarrow |V_M - L| < \varepsilon.
\]

\( \therefore \quad \limsup S_n = \lim_{M \to \infty} V_M = L \).

Similarly \( \liminf S_n = \lim_{M \to \infty} \inf \{ S_n | n > M \} = L \).

(ii) Let \( V_M = \sup \{ S_n | n > M \}, \quad U_M = \inf \{ S_n | n > M \} \)

Then \( \forall n > M \)
\[
U_M \leq S_n \leq V_M
\]

Since \( \liminf S_n = \limsup S_n = L \neq \pm \infty \)

Thus \( \forall \varepsilon > 0 \) \( \exists M \) \( \forall n > M \)
\[
|V_M - L| < \varepsilon \quad \text{and} \quad |U_M - L| < \varepsilon
\]
\[
\Rightarrow L - \varepsilon < \inf S_n < S_n < V_M < L + \varepsilon \quad \forall n > M
\]
\[
\Rightarrow L - \varepsilon < S_n < L + \varepsilon \quad \forall n > M
\]
\[
\Rightarrow |S_n - L| < \varepsilon
\]
\[
\Rightarrow \lim_{n \to \infty} S_n = L.
\]

\( \therefore \quad S_n = (-1)^n \quad V_M = \sup \{ (-1)^n | n > M \} = +1 \)
\[
U_M = \inf \{ (-1)^n | n > M \} = -1
\]
\[
\Rightarrow \lim \inf (-1)^n = -1 \neq 1 = \lim \sup (-1)^n
\]
\[ \lim_{n \to \infty} (-1)^n \text{ does not exist.} \]

Cauchy sequences.

Definition A sequence \( \{s_n\} \) of real numbers is Cauchy if \( \forall \varepsilon > 0 \ \exists N \text{ so that} \]
\[ n, m > N \implies |s_n - s_m| < \varepsilon. \]

Why the definition?

1) Theorem 10.11 A sequence is Cauchy \iff it converges.

\[ \text{the definition provides another way to prove/check that} \]
\[ \text{a sequence converges without knowing what it converges to} \]

2) One can define/construct \( \mathbb{R} \) out of \( \mathbb{Q} \) as

equivalence classes of Cauchy sequences in \( \mathbb{Q} \).

Proof of 10.11

(\( \Rightarrow \)) Suppose \( \lim_{n \to \infty} s_n = L \). Then \( \forall \varepsilon > 0 \ \exists N \text{ so that} \]
\[ n > N \implies |s_n - L| < \frac{\varepsilon}{2}. \]

\[ \implies \forall n, m > N \]
\[ |s_n - s_m| = |s_n - L + L - s_m| \leq |s_n - L| + |L - s_m| \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

(\( \Leftarrow \)) Suppose \( \{s_n\} \) is a Cauchy sequence.

We first prove

Lemma 10.10 Cauchy sequences are bounded.

\[ \text{Proof: let } \{s_n\} \text{ be a Cauchy sequence.} \]

Then \( \exists N \in \mathbb{N} \)
\[ n, m > N \implies |s_n - s_m| < 1 \]
\[ \implies \forall m > N \]
\[ |s_n - s_m| < 1 \]
\[ |S_n| \leq |S_n - S_m| + |S_m| < 1 + |S_n| \quad \forall m > N. \]

\[ |S_n| < \max \{|S_n|, |S_{n+1}|\}. \]

Now we argue: \(|S_n|\) Cauchy \(\Rightarrow\) \(\lim \sup S_n = \lim \inf S_n\)

We'll do it next time...