Last time, Thm 25.2: \( \{ f_n \} \in C^0([a,b]), f_n \rightarrow f \) uniformly.

Then \( \left. \int_a^b f_n \right| \rightarrow \int_a^b f \).

Corl 33.1: Suppose \( \sum_{n=0}^{\infty} f_n(x) \) converges uniformly to \( f \) on \( [a,b] \).

Then \( \int_a^b f = \sum \left( \int_a^b f_n(x) \right) \).

Thm 26.4: Suppose \( f(x) = \sum a_n x^n \) has radius of convergence \( R > 0 \).

Then \( \forall x \in (-R,R) \),
\[
\int_0^x \left( \sum_{n=0}^{\infty} a_n t^n \right) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.
\]

Thm 233.2: Suppose \( \{ f_n \} \in C^1([a,b]) \), \( f_n \rightarrow f \) on \( [a,b] \) and 
\( f_n' \rightarrow g \) uniformly on \( [a,b] \). Then \( f \) is differentiable and \( f' \rightarrow g \) (\( = g \)).

Lemma 26.3: Suppose \( \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence \( R \).

Then the power series
\[
\sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}
\]
also have radius of convergence \( R \).

Proof:
\[
\sum_{n=0}^{\infty} n a_n x^n = x (\sum_{n=0}^{\infty} n a_n x^n) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = x \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n
\]

Recall (Thm 9.7(c)) \( \lim n^{\frac{1}{n}} = 1 \)

\( \lim \sup |n a_n|^{\frac{1}{n}} = \lim n^{\frac{1}{n}} \lim \sup |a_n|^{\frac{1}{n}} = \lim \sup |a_n|^{\frac{1}{n}} = \frac{1}{R} \).

(Thm 12.1)

\( \implies \sum n a_n x^n \) has radius of convergence \( R \).

For the second series we use the fact that \( \lim_{n \to \infty} (n+1)^{\frac{1}{n+1}} = 1 \) and consequently
\( \lim_{n \to \infty} (n+1)^{\frac{1}{n}} = 1 \) as well.
Theorem 26.5. Suppose \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence \( R \). Then \( f \) is differentiable on \(( -R, R)\) and 
\[
 f'(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}
\] for \( x \in (-R, R) \).

Proof (different from text). By 26.3 \( g(x) = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} \) has radius of convergence \( R \).

Consider \( f_n(x) = \sum_{k=0}^{n} a_k x^k \). \( f_n'(x) = \sum_{k=0}^{n} k \cdot a_k x^{k-1} \) converges uniformly to \( f \) on \([-R', R'] \) for \( R' < R \).

By 33.2 \( f_n \) is differentiable on \([-R', R'] \) and 
\[
 f'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1}
\] for \( x \in (-R, R) \) and 
\[
 f'(0) = \sum_{n=0}^{\infty} a_n \cdot 0^{n-1} = a_0
\]

Corollary. Any power series with positive radius of convergence is infinitely differentiable \( (C^\infty) \).

Proof. Induction. Any power series is differentiable, and its derivative has the same radius of convergence. Now differentiate again....

Def 31.2. Suppose \( f \in C^\infty((-\epsilon, \epsilon)) \) for some \( c \in \mathbb{R}, \epsilon > 0 \).

The Taylor series of \( f \) is the power series 
\[
 Taylor(f,c) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c) \cdot (x-c)^k.
\]

Remark. Suppose \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) has a positive radius of convergence \( R \). Then

\[
 Taylor(f,0) = \sum_{n=0}^{\infty} a_n x^n
\]

Reason \( f'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1} \) so \( f'(0) = 1 \cdot a_1 \)
\[
 f''(x) = \sum_{n=0}^{\infty} n(n-1) \cdot a_n x^{n-2} \) so \( f''(0) = 2 \cdot 1 \cdot a_2 \)
Induction on $k \Rightarrow f^{(k)}(x) = \sum_{n=0}^{\infty} a_n (n-1) \ldots (n-k) a_n x^{n-k}$

and

$f^{(k)}(0) = k! a_k$

Hence

\[ \text{Taylor } (f, 0) = \sum_{k=0}^{\infty} \frac{1}{k!} (k! a_k) x^k = \sum_{k=0}^{\infty} a_k x^k \]

Abel's theorem (266) Let $f(x) = \sum a_n x^n$ be a power series with a finite positive radius of convergence $R$. If the series converges at $x = R$ then it converges uniformly on $[0, R]$. Similarly if it converges at $x = -R$, it converges uniformly on $[-R, 0]$.

Consequently if $f$ converges at $R$ it's continuous at $R$; if $f$ converges at $-R$, it's continuous at $-R$.

Proof. We consider first special case: $R = 1$, series converges at $x = 1$.

We want to apply Cauchy's criterion: given $\varepsilon > 0$ we want to find $N$ such that $n \geq m > N \Rightarrow$

\[ |a_m x^m + \ldots + a_n x^n| < \varepsilon \quad \forall x \in [0, 1] \]

By subtracting $f(1)$ from $f(x)$ we may assume $f(1) = 0$.

Let $f_n(x) = \sum_{k=0}^{n} a_k x^k, \quad S_n = \sum_{k=0}^{n} a_k$ (note $f_n(1) = S_n$)

\[ |f_n(x) - f_{m-1}(x)| = |a_m x^m + \ldots + a_n x^n| \]

Now

\[ S_k - S_{k-1} = (a_0 + \ldots + a_k) - (a_0 + \ldots + a_{k-1}) = a_k \]

\[ \Rightarrow |f_n(x) - f_{m-1}(x)| = \left| \sum_{k=m}^{n} (S_k - S_{k-1}) x^k \right| \]

\[ = \left| \sum_{k=m}^{n} S_k x^k - x \sum_{k=m}^{n} S_{k-1} x^{k-1} \right| = \left| \sum_{m}^{n} S_k x^k - x \sum_{m-1}^{n-1} S_k x^k \right| \]

\[ = \left| S_m x^m - S_{m-1} x^m + (1-x) \sum_{k=m}^{n-1} S_k x^k \right| \]
Since \( \lim_{n \to \infty} s_n = f(1) = 0 \), given \( \varepsilon > 0 \) \( \exists N\in\mathbb{N} \) such that \( |s_n| < \varepsilon/3 \) for all \( n \geq N \).

For \( n \geq m \geq N \), \( x \in [0,1) \)

\[
| (1-x) \sum_{k=m}^{n} s_k x^k | \leq \frac{\varepsilon}{3} \cdot \left( 1-x \right) \sum_{k=m}^{n} x^k
\]

\[
= \frac{\varepsilon}{3} \left( 1-x \right) \cdot \left( \frac{1-x^{m+1}}{1-x} - \frac{1-x^m}{1-x} \right)
\]

\[
= \frac{\varepsilon}{3} \left( x^m - x^{m+1} \right) < \frac{\varepsilon}{3}.
\]

If \( x = 1 \), \( | (1-x) \sum_{m}^{n} s_k x^k | = 0 < \frac{\varepsilon}{3} \).

\[ \Rightarrow \quad \text{For } n \geq m \geq N, \quad x \in [0,1] \]

\[
| f_n(x) - f_m(x) | \leq |s_n x^n| + |s_m x^m| + \frac{\varepsilon}{3}
\]

\[
< \frac{\varepsilon}{3} \cdot 1 + \frac{\varepsilon}{3} \cdot 1 + \frac{\varepsilon}{3} = \varepsilon.
\]

**Case 2**. \( f(x) \) has radius of convergence \( R \) and converges at \( x = R \).

Let \( g(x) = f(Rx) = \sum a_n R^n x^n \).

Then \( g \) has radius of convergence \( R \) and converges at \( x = 1 \) by case 1. \( g \) converges uniformly on \( C(0,1) \).

By case 1, \( f(x) = g \left( \frac{x}{R} \right) \) converges uniformly on \( C(0,R) \).

**Case 3**. \( f(x) \) has radius of convergence \( R \), and converges at \( x = -R \).

Let \( h(x) = f(-x) \).

By case 2, \( h \) converges uniformly on \( C(0,R) \).

\( f(x) = h(-x) \) converges uniformly on \( C(-R,0) \).