Last time: If \( f: [a,b] \to \mathbb{R} \) is bounded then
\[ L(f) \leq U(f), \]
where
\[ L(f) = \sup_{P} \int f(P), \quad U(f) = \inf_{P} \int f(P), \quad \text{etc.} \]
- Proved from the definition: \( \int_{a}^{b} x \, dx = \frac{1}{2} b^2 \).

Example: Suppose \( f: [a,b] \to \mathbb{R} \) is bounded and integrable (\( L(f) = U(f) \))
and \( g: [a,b] \to \mathbb{R} \) agrees with \( f \) except at finitely many points.
Then \( g \) is integrable and \( \int_{a}^{b} g = \int_{a}^{b} f \).

Proof: It's enough to consider the case: \( g(x) = f(x) \) for all \( x \in [a,b] \setminus \{x_0, x_1, \ldots, x_n\} \).
Since \( f \) is integrable, \( U(f) = L(f) \). Choose a sequence of partitions \( P_n \) so that \( U(f, P_n) \to U(f) \).
We assume: \( x_0 \in (a,b) \) (the case where \( x_0 = a \) or \( x_0 = b \) is similar).
For \( n \) large let \( Q_n = P_n \cup \{x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\} \).
Since \( Q_n \geq P_n \), \( U(f, Q_n) \leq U(f, P_n) \).
For \( x \in [a,b] \),
\[ g(x) \leq f(x) + \frac{1}{b} |g(x_0) - f(x_0)|. \]
\[ \Rightarrow \sup_{Q_n} g(x) \leq \sup_{P_n} f(x) + \frac{1}{b} |g(x_0) - f(x_0)|. \]
\[ \Rightarrow U(g, Q_n) \leq U(f, Q_n) + \frac{1}{b} \left| g(x_0) - f(x_0) \right|. \]
\[ \Rightarrow U(g) \leq U(g, Q_n) \leq U(f, Q_n) + \frac{2}{b} \left| g(x_0) - f(x_0) \right|. \]
\[ \Rightarrow U(g) \leq U(f) \quad \text{as } n \to \infty \]
\[ \Rightarrow U(g) = U(f) \]
Similarly, \( L(g) \geq L(f) \).
Therefore \( U(f) = L(f) \leq L(g) \leq U(g) \leq U(f) \).

Cauchy criterion for integrability (Thm 32.5): \( f: [a,b] \to \mathbb{R} \) is bounded
\( \Rightarrow \) integrable on \( [a,b] \) \( \Rightarrow \) \( \forall \varepsilon > 0 \) \( \exists \) \( D \) partition \( P \) of \( [a,b] \) s.t.
\[ U(f, P) - L(f, P) < \varepsilon \]
Proof (\(\Rightarrow\)) Since \(f\) is integrable, \(L(f) = U(f)\). Given \(\varepsilon > 0\)

There exist partitions \(P_1, P_2\) such that

\[
U(f, P_1) \leq U(f) + \varepsilon/2, \quad L(f, P_2) \geq L(f) - \varepsilon/2
\]

Let \(P = P_1 \cup P_2\). Then

\[
U(f, P) \leq U(f, P_1) \leq U(f) + \varepsilon/2 \quad \Rightarrow \quad U(f, P) - L(f, P) \leq \frac{U(f) - L(f) + \varepsilon}{2}
\]

\(\Rightarrow\) Suppose \(\varepsilon > 0\) and \(P\) such that \(U(f, P) - L(f, P) < \varepsilon\). Then

\[
U(f) \leq U(f, P) = U(f, P) - L(f, P) + L(f, P) \leq L(f, P) + \varepsilon \leq L(f) + \varepsilon
\]

\(\Rightarrow\) \(\exists \varepsilon' > 0\) such that \(U(f) - L(f) \leq \varepsilon'\). \(\Rightarrow\) \(U(f) - L(f) = 0\). \(\blacksquare\)

Riemann integrability

We need a few more definitions.

**Def** The **mesh** of a partition \(P: t_0 < t_1 < \cdots < t_n\) is the maximum length of the intervals \([t_i, t_{i+1}]\), \(i = 0, \ldots, n-1\):

\[
\text{mesh}(P) = \max \{ t_{i+1} - t_i \} \quad i = 0, 1, \ldots, n-1.
\]

**Def** Let \(f: [a, b] \rightarrow \mathbb{R}\) be a bounded function and \(P = \{t_0, \ldots, t_n\}\) a partition of \([a, b]\). A **Riemann sum** of \(f\) associated with \(P\) is the sum of the form

\[
S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})
\]

where \(x_k \in [t_{k-1}, t_k]\); the choice of \(x_k\)'s is arbitrary.

**Def** A function \(f: [a, b] \rightarrow \mathbb{R}\) is **Riemann integrable** if \(\exists R \in \mathbb{R}\) so that \(\forall \varepsilon > 0\) \(\exists \delta > 0\) so that a partition \(P\) with \(\text{mesh}(P) < \delta\) and a Riemann sum \(S\) of \(f\) associated with \(P\) we have

\[
|S - R| < \varepsilon.
\]

\(R\) is called the **Riemann integral** of \(f\).
Theorem 32.9. A bounded function \( f: [a,b] \to \mathbb{R} \) is Riemann integrable \( \iff \) it is (Darboux)-integrable. The values of the two integrals agree.

We may prove this theorem later. From now on we will not distinguish between Darboux and Riemann integrals.

Properties of integrals

Theorem 33.1. Every monotonic function \( f: [a,b] \to \mathbb{R} \) is integrable.

Proof. We assume \( f \) is non-decreasing (if \( f \) is non-increasing the proof is similar).

Since \( \forall x \in [a,b), \ f(a) \leq f(x) \leq f(b), \ f \) is bounded.

We'll prove integrability of \( f \) using Cauchy criterion.

Let \( P_n \) be the partition that divides \( [a,b] \) into \( n \) equal pieces:

\[
P_n = t_0 < t_1 < \ldots < t_n, \quad t_k = a + \frac{b-a}{n} \cdot k
\]

Then

\[
U(f, P_n) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} f(t_k) \cdot \frac{b-a}{n}
\]

\[
L(f, P_n) = \sum_{k=1}^{n} f(t_{k-1}) \cdot \frac{b-a}{n}.
\]

\[
U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \cdot \sum_{k=1}^{n} f(t_k) - f(t_{k-1})
\]

\[
= \frac{b-a}{n} \left( (f(t_2) - f(t_1)) + (f(t_3) - f(t_2)) + \ldots + (f(t_n) - f(t_{n-1})) \right)
\]

\[
= \frac{b-a}{n} \left( f(b) - f(a) \right)
\]

Now given \( \varepsilon > 0 \), choose \( n \) so that \( \frac{(b-a)(f(b) - f(a))}{n} < \varepsilon \).

Then \( U(f, P_n) - L(f, P_n) < \varepsilon \). \( \square \)
Theorem 33.2  Every continuous function \( f \) on \([a, b]\) is integrable.

Proof: Since \( f: (a,b) \to \mathbb{R} \) is continuous and \([a, b]\) is compact, \( f \) is uniformly continuous: \( \forall \varepsilon > 0 \ \exists \delta > 0 \) s.t. 
\[ |x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a} \ \forall \ x, y \in [a, b]. \]

Let \( P \) be a partition with \( \text{mesh}(P) < \delta \).

For each \( k, \exists x_k, y_k \in [t_{k-1}, t_k] \) so that
\[
\begin{align*}
f(x_k) &= \sup \{ f(x) \mid x \in [t_{k-1}, t_k] \} = M(f, [t_{k-1}, t_k]) \\
f(y_k) &= \inf \{ f(x) \mid x \in [t_{k-1}, t_k] \} = m(f, [t_{k-1}, t_k])
\end{align*}
\]
\[ \Rightarrow m(f, [t_{k-1}, t_k]) - M(f, [t_{k-1}, t_k]) = f(x_k) - f(y_k) < \frac{\varepsilon}{b-a}. \]

Since \( |x_k - y_k| \leq |t_k - t_{k-1}| < \delta \)

Therefore
\[
U(f, P) - L(f, P) = \sum_{k=1}^{n} (M(f, [t_k, t_{k-1}]) - m(f, [t_k, t_{k-1}]))(t_k - t_{k-1})
\leq \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \frac{\varepsilon \cdot (b-a)}{b-a} = \varepsilon.
\]

By Cauchy criterion \( f \) is integrable. \( \square \)

Theorem 33.3  Let \( f \) and \( g \) be two integrable functions on \([a, b]\). Then

i) if \( c \in \mathbb{R} \) \( cf \) is integrable and \( \int_{a}^{b} cf = c \int_{a}^{b} f \)

ii) \( f + g \) is integrable and \( \int_{a}^{b} (f + g) = \int_{a}^{b} f + \int_{a}^{b} g \)

Hence \( \mathcal{S}_{a}^{b} \) : \{ integrable functions on \([a, b]\) \} form a vector space and,

a) \( \mathcal{S}_{a}^{b} \): \{ integrable functions \} \to \mathbb{R} \) is linear.