HW #1 posted. Due Sep 3, 2014, in class

- Dedekind-Peano axioms for \( \mathbb{N} \)
- definition of an abelian group
- definition of a commutative ring
- definition of a field

\( \mathbb{Q}, \mathbb{R} \) are fields, \( \mathbb{Z} \) is a ring, not a field.

\[ a \in \mathbb{Z}, \, a \neq \pm 1 \text{ has no multiplicative inverse} \]

\[ \mathbb{F}_2 \neq \mathbb{Q} \]

Today:
- ordered fields
- completeness axiom for \( \mathbb{R} \)

### Definition

A relation \( \leq \) on a set \( S \) is a linear order (total order) if

1. \( a, b, c \in S \), \( a \leq b \) and \( b \leq c \) \( \Rightarrow \) \( a \leq c \) (antisymmetry)
2. \( a, b, c \in S \), \( a \leq b \) and \( b \leq c \) \( \Rightarrow \) \( a \leq c \) (transitivity)
3. \( a, b \in S \), either \( a \leq b \) or \( b \leq a \) (totality)

\((S, \leq)\) is called an ordered set.

- Ex. \( S \subseteq \mathbb{Z} \) \( \leq \) is the usual \( \leq \)
- Ex. \( S = \mathbb{R}^2 \), \((x_1, y_1) \leq (x_2, y_2)\) if \( x_1 < x_2 \) or \((x_1 = x_2 \text{ and } y_1 \leq y_2)\) "lexicographic order".

### Definition

An ordered field \( \mathbb{F} \) is a field \((\mathbb{F}, +, 0, \cdot, 1)\) together with a linear order \( \leq \) so that

1. \( (O^+) \) if \( a \leq b \), then \( a + c \leq b + c \) for all \( a, b, c \in \mathbb{F} \)
2. \( (O^*) \) if \( 0 \leq a \) and \( 0 \leq b \), then \( 0 \leq ab \)

Example: \((\mathbb{Q}, \leq)\) and \((\mathbb{R}, \leq)\) are ordered fields.

Note: Instead of \((O^*)\), the text requires

\[ (O^*) \quad a \leq b \text{ and } a \cdot c = ac \leq bc. \]

Non-Ex. \( \mathbb{F} = \mathbb{R}^2 \) with lexicographic order is not an ordered field.
Lemma: (05) and (05') are equivalent.

We first prove (05).

Theorem 3.1 (ii) For any field \( F \), for any \( a \in F \)
\[ 0 \cdot a = 0. \]

Proof:
\[
0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a. \quad \Rightarrow
\]
\[
0 = 0 \cdot a + (-0 \cdot a) = 0 \cdot a + 0 \cdot a + (-0 \cdot a)
\]
\[
= 0 \cdot a + 0 = 0.9. \quad \blacksquare
\]

Proof of (05) and (05') are equivalent:

\((05 \Rightarrow 05')\). If \( 0 \leq a \) and \( 0 \leq b \), then \( 0 \leq a \cdot b \) by 05.

By Thm 3.1 (ii) \( 0 \cdot b = 0 \). Hence \( 0 \leq a \cdot b \).

\[(05' \Rightarrow 05)\]
Suppose \( a \leq b \). Then \( 0 = a + (-a) \leq b + (-a) \)

\[\text{(07')}\]
\[
0 \leq b + (-a).c = b \cdot c + (-a) \cdot c
\]
\[
\Rightarrow 0 + a \cdot c \leq b \cdot c + (-a) \cdot c + a \cdot c
\]
\[
\Rightarrow a \cdot c \leq b \cdot c + (-a + a) \cdot c
\]
\[
= a \cdot c \leq b \cdot c + 0 \cdot c = b \cdot c. \quad \blacksquare
\]

One can prove all sorts of inequalities for ordered fields that one normally takes for granted.

Thm 3.2 (ii) For any ordered field \( F \) and \( a, b \in F \)
\[ a \leq b \Rightarrow -b \leq -a. \]

Proof:
\[
a \leq b \Rightarrow a + (-a) + (-b) \leq b + (-a) + (b)
\]
\[
\Rightarrow (a + (-a)) + (-b) \leq (-a) + (b + (-b))
\]
\[
\Rightarrow 0 + (-b) \leq (-a) + 0
\]
\[
\Rightarrow -b \leq -a. \quad \blacksquare
\]

There are more things to prove.
Read Thm 3.1 and Thm 3.2.
Absolute value. Let \((F, \leq)\) be an ordered field. We define

\[ |a| = \begin{cases} 
  a & \text{if } a \geq 0 \\
  -a & \text{if } a \leq 0 
\end{cases} \]

Intuition: \(|a|\) is the distance from \(a\) to 0.

\[ \overbrace{\overbrace{b}^{a}}^{a} \quad \overbrace{\overbrace{a}^{a}}^{b} \]

More generally:

**Def** The distance between \(0, b \in F\) is \(|b-a|\).

\[ \text{dist}(a, b) := |b-a| \]

**Theorem 3.5**

(i) \(|a| \geq 0\) for all \(a\)

(ii) \(|ab| = |a||b|\) for all \(a, b\)

(iii) \(|a+b| \leq |a| + |b|\) for all \(a, b\).

**Proof:** see textbook.

**Corollary 3.6** (triangle inequality) For all \(a, b, c\),

\[ \text{dist}(a, c) \leq \text{dist}(a, b) + \text{dist}(b, c) \]

**Proof of corollary**

\[ \text{dist}(a, c) = |c-a| = |c-b+b-a| \leq |c-b| + |b-a| = \text{dist}(b, c) + \text{dist}(a, b). \]

**Useful facts**

(i) \(|b| \leq a \iff -a \leq b \leq a\)

(ii) \(|a|-|b| \leq |a-b|\)

**Proof** homework.
Def Suppose \((S, \leq)\) is an ordered set, \(E \subseteq S\) a subset. \(\beta \in S\) is an upper bound for \(E\) if 
\[ x \leq \beta \quad \text{for all } x \in E. \]

\[ \exists x \quad (S, \leq) = (\mathbb{Q}, \leq) \]
\[ E = \{ x \in \mathbb{Q} \mid x^2 \leq 2.5 \} = \{ x \in \mathbb{Q} \mid -1.5 < x < 1.5 \}. \]

Then \(3\) is an upper bound for \(E\).

So \(2.5\) (since \(x < 1.5 \Rightarrow x^2 \leq 1.5^2 = 2.25\))
and if \(x^2 \leq 2\) then \(x^2 \leq 2.25\)

So is \(1.45\). . .

Def If \(E \subseteq (S, \leq)\) has an upper bound we say that \(E\) is bounded above.

\[ \exists x \quad (-\infty, 1] \subseteq \mathbb{R}\] is bounded above,
\[ (0, \infty) \subseteq \mathbb{R}\] is not bounded above.

Similarly one defines lower bound and bounded below.

Def (Least upper bound) Let \((S, \leq)\) be an ordered set, \(E \subseteq S\) a subset. \(\beta \in S\) is a least upper bound of \(E\) if

(i) \(\beta\) is an upper bound of \(E\)

(ii) if \(\beta\) is any other upper bound of \(E\), \(\beta \leq \beta\).

Note: least upper bounds are unique: if \(\beta_0, \beta_1\) are two least upper bounds of \(E\) then
\[ \beta_0 = \beta_1, \quad \text{and} \quad \beta_1 \leq \beta_0 \]

\[ \Rightarrow \beta_0 = \beta_1. \]

We write \(\text{Sup } E\) for the least upper bound. (If it exists!)
Completeness axiom for \( \mathbb{R} \):

Any nonempty subset \( E \) of \( \mathbb{R} \) bounded above has the least upper bound.

Note: This is not true for \( \mathbb{Q} \):

\[ E = \{ x \in \mathbb{Q} \mid x^2 \leq 2 \} \] is bounded above but \( E \) \( \notin \) \( \mathbb{Q} \); so while there are lots of upper bounds, there are no least ones.