1 Consider the saddle \( z = xy \) parametrized as a graph. Compute the first and second fundamental forms \((g_{ij})\) and \((h_{ij})\). Show that the Gauss and mean curvatures of the saddle \( z = xy \) are given by

\[
K(x, y, z) = \frac{-1}{(1 + x^2 + y^2)^2}, \quad H(x, y, z) = \frac{-z}{(1 + x^2 + y^2)^{3/2}}.
\]

2 a Suppose \( y = f(x), x \in (-R, R) \), is a smooth function with \( f(0) = 0 \) and \( f(x) \geq R + \sqrt{R^2 - x^2} \) for some number \( R > 0 \). Prove that \( f'(0) = 0 \) and \( f''(0) > \frac{1}{R} \). What can you say about the curvature of the curve \( \gamma(t) = (t, f(t)) \) at \( t = 0 \)?

b Suppose \( z = f(x, y), x^2 + y^2 < R^2 \), is a smooth function with \( f(0, 0) = 0 \) and

\[
f(x, y) \geq R + \sqrt{R^2 - x^2 - y^2}
\]

for some number \( R > 0 \). Let \( S \) be the graph of \( f \). Prove that for any vector \( X \) of length 1 tangent to \( S \) at \( p = (0, 0, 0) \) we have

\[
II_p(X, X) \geq \frac{1}{R}.
\]

Here as usual \( II_p \) is the second fundamental form of \( S \) at \( p \).

c Explain how part b above is used to prove that any compact surface has a point where the Gauss curvature is positive.

3 Suppose \((a_{ij})\) is a symmetric \(2 \times 2\) matrix with eigenvalues \( \kappa_1 < \kappa_2 \). Prove that the corresponding eigenvectors \(X_1, X_2\) are perpendicular to each other. What does this tell you about the eigenvectors of the Weingarten map?

4 Suppose \( F : U \to \mathbb{R}^3 \) is a parametrization of a regular surface \( S \) and suppose that at a point \( p \in S \) we have

\[
W_p(\frac{\partial F}{\partial u^1}) = 3 \frac{\partial F}{\partial u^1}, \quad W_p(\frac{\partial F}{\partial u^2}) = -5 \frac{\partial F}{\partial u^1} - 2 \frac{\partial F}{\partial u^2},
\]

where, as usual, \( W_p \) denotes the Weingarten map at \( p \).

a Show that \( \frac{\partial F}{\partial u^1} \) and \( \frac{\partial F}{\partial u^2} + \frac{\partial F}{\partial u^1} \) are both principal directions at \( p \).

b What are the principal, Gauss and mean curvatures of \( S \) at \( p \)? Explain.

5 Let \( z = f(x, y) \) be a smooth function whose first partials vanish at a point \((u, v) \in \mathbb{R}^2\) and the second partials satisfy

\[
\frac{\partial^2 f}{\partial x \partial y}(u, v) = 0, \quad \frac{\partial^2 f}{\partial x^2}(u, v) = a, \quad \frac{\partial^2 f}{\partial y^2}(u, v) = b.
\]

What are the principal curvatures of the graph of \( f \) at the point \((u, v, f(u, v))\)? Explain. What theorem are you using?
6. Let $S$ be a regular surface. Prove that for any point $p \in S$ there is a parametrization $F : U \to S \cap V$ with $F(q) = p$ and all Christoffel symbols $\Gamma^k_{ij}(q) = 0$. Hint: 3.6.15 and 4.2.14.

7. Suppose $S$ is a surface of finite area such that for any family $\{S_t\}$ of surfaces with $S_0 = S$ we have

$$\text{Area}(S) \leq \text{Area}(S_t).$$

What can you say about the mean, Gauss and principal curvatures of $S$?

8. Prove that a minimal surface cannot be compact (saying “this is theorem blah in Bär” is not a proof).

9. Let $X = (-y, x, z), Y = (x, y, z)$ be two vector fields on $\mathbb{R}^3$. Compute the covariant derivatives $D_XY, D_YX$ and the Lie bracket $[X,Y]$.

10. A vector field $X$ in $\mathbb{R}^3$ is tangent to a surface $S$ if $X_p \in T_pS$ for every point $p \in S$. Prove that if vector fields $X$ and $Y$ are tangent to $S$ then so is their Lie bracket.

   Hint: recall that for any point $p \in S$ there is an open neighborhood $V$ of $p$, an open neighborhood $W$ of $(0,0,0)$ and a change of coordinates $\varphi : V \to W$ so that

   $$\varphi(S \cap V) = W \cap \{ z = 0 \}.$$

   (Lecture 16 on 10/3). Therefore it is enough to prove the result for $S = \{ z = 0 \}$.