1 Prove that if $c : [a, b] \rightarrow S$ is a geodesic on a regular surface $S$ then $c$ is parametrized proportionally to arc length.

Prove that given any regular curve $\gamma : [a, b] \rightarrow S$ there is a change of parameter so that the new curve is parametrized by arc length. Does that mean that any regular curve can be turned into a geodesic by a change of parameter? Explain.

2 Show that $S = \{(x, y, z) \mid e^{xy} - 2 \cos z = 0\}$ is a regular surface.

Show that the vector field $X = (ye^{xy}, xe^{xy}, 2 \sin z)$ is normal to $S$.

Show that the vector field $W = (-x, y, 0)$ is tangent to $S$.

Compute $D_W W$ and $\nabla W W$.

3 Consider the surface $S$ parametrized by

$$F(u, v) = (u, u^2 \cos v, u^2 \sin v), \quad u > 0, v \in \mathbb{R}.$$ 

Compute the corresponding first fundamental form, the second fundamental form, the Gauss and mean curvatures of $S$.

4 Give an example of a regular surface $S$ and of a Riemannian metric $g$ on $S$ that could not possibly be the first fundamental form of $S$ no matter how $S$ sits in $\mathbb{R}^3$. Explain why.

5 Let $c : [a, b] \rightarrow \mathbb{R}^3$ be a regular curve. Define its curvature $\kappa$ and torsion $\tau$. What is the Frenet frame of $c$? How is it defined? What are the Frenet equations?

5b Consider the curve $\gamma(t) = (a \cos \frac{t}{c}, a \sin \frac{t}{c}, b \frac{t}{c^2})$, where $a, b, c$ are positive constants with $c^2 = a^2 + b^2$.

Show that $\gamma(t)$ has unit speed. Compute the curvature and torsion of $\gamma(t)$.

6 Suppose $c_1, c_2 : [a, b] \rightarrow \mathbb{R}^3$ are two regular curves with the same curvature and torsion. Suppose further that $c_1(a) = c_2(a)$ and $\dot{c}_1(a) = \dot{c}_2(a)$. Does it follow that $c_1(t) = c_2(t)$? Explain. If not, what can you say about the two curves?

7 Define what it means for $c : [a, b] \rightarrow \mathbb{R}^2$ to be simple, to be closed.

Suppose $c : [a, b] \rightarrow \mathbb{R}^2$ is a simple closed curve of length $\ell$ enclosing a region $D$. What do you know about the area of $D$? Can it be arbitrarily large? arbitrarily small?

8 Define an oriented surface. Define an orientable surface. What’s the difference between the two concepts?

9 Suppose the surface $S$ is oriented by a unit normal $N$. Suppose further that the Gauss curvature of $S$ at $p \in S$ is $k$ and the mean curvature of $S$ at $p$ is $h$. What are the Gauss and mean curvatures of $S$ at $p$ if $S$ is oriented by $-N$?
10 What is the mean curvature of the cylinder $S = \{(x, y, z) \mid x^2 + y^2 = 1\}$?

11 Let $S$ be a regular surface with a Riemannian metric $g$ and let $c : [a, b] \to S$ be a regular curve. What is a vector field on $S$ along $c$? What does it mean for such a vector field to be parallel?

12 Prove that if a vector field $v : [a, b] \to \mathbb{R}^3$ on $S$ along $c$ is parallel then

$$g_{c(a)}(v(a), v(a)) = g_{c(b)}(v(b), v(b)).$$

Hint: show that the function $f(t) = g_{c(t)}(v(t), v(t))$ is constant.

13 What is the definition of Riemann curvature tensor $R$ of a connection $\nabla$? What can you say about the Riemann curvature tensor of the cylinder $\{x^2 + y^2 = 1\}$ of the plane $\{z = 1\}$?

14 What does it mean for a surface to be compact? Can the Gauss curvature of a compact surface be negative at every point of the surface?

15 Suppose that $S$ is a regular surface and suppose that the image of the Gauss map $N : S \to S^2$ traces out a circle on $S^2$. Prove that $S$ is not compact.