1. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T(1,0) = (4,3)$ and $T(0,1) = (-2,-1)$. Find the eigenvalues and eigenvectors of $T$. Are the eigenvectors orthogonal to each other?

2. Let $S$ be an oriented surface and $p \in S$ a point. Denote the normal curvature of $S$ at $p$ in the direction of $X \in T_p S$, $||X|| = 1$, by $k(X)$. That is, $k(X) := II_p(X,X)$. We thus get a map $k : \{X \in T_p S \mid ||X|| = 1\} \to \mathbb{R}$. Show that the maximum and minimum values of $k$ are principal curvatures. Hint: Euler’s formula: if $X_1, X_2$ are principal curvature directions, and $X = \cos \varphi X_1 + \sin \varphi X_2$ then $k(X) = \ldots$.

3. Let $S$ be the saddle $z = x^2 - y^2$. Find the normal curvature $k(1/\sqrt{2},1/\sqrt{2},0)$ to $S$ at the point $p = (0,0,0)$ (same notation as in problem 2).

4. Consider a curve $c(t) = (g(t), h(t))$ in the $xy$-plane with no self-intersections and with $h(t) > 0$ for all $t$. Rotating this curve around the $x$ axis in 3-space gives rise to a surface of revolution $S$. Its parametrization is given by $F(u,v) = (g(u), h(u) \cos v, h(u) \sin v)$, where we should either assume that $v \in (0,2\pi)$ or that $v \in (-\pi, \pi)$.

   a. Show that $N(u,v) = (h'(u), -g'(u) \cos v, -g'(u) \sin v) / \sqrt{(g'(u))^2 + (h'(u))^2}$ is a unit normal field to the surface $S$.

   b. Show that the first fundamental form of $S$ with in the coordinates given by the parametrization above is $((g')^2 + (h')^2) \, du \otimes du + h^2 \, dv \otimes dv$ and the second fundamental form is

   $$\frac{(g''h' - h''g')}{\sqrt{(g')^2 + (h')^2}} \, du \otimes du + \frac{hg'}{\sqrt{(g')^2 + (h')^2}} \, dv \otimes dv.$$

   c. Show that the Gauss curvature is

   $$K = \frac{g'(g''h' - h''g')}{{h((g')^2 + (h')^2)}}.$$

   d. Show that $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ are eigenvectors of the Weingarten map. What are the lines of curvature? Hint: you have already computed the first and second fundamental forms in coordinates $u, v$. Therefore you should be able to reconstruct the matrix for the Weingarten map from the matrices for the first and second forms.