Given a regular space curve $c : I \to \mathbb{R}^3$ it is common to refer to the collection $\kappa, \tau, v, n, b$ (where $\kappa$ is curvature, $\tau$ is torsion, $v$ is velocity, $n$ unit normal and $b$ unit binormal) as the Frenet apparatus.

1. Compute the Frenet apparatus $\kappa, \tau, v, n, b$ for the unit speed curve
   
   \[ c(t) = \left( \frac{4}{5} \cos t, 1 - \sin t, -\frac{3}{5} \cos t \right). \]

   Show that the curve is a circle; find its center and radius.

2. Consider the curve

   \[ \beta(s) = \left( \frac{(1 + s)^{3/2}}{3}, \frac{(1 - s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right) \]

   defined on the interval $I = (-1, 1)$. Show that $\beta$ has unit speed, and compute its Frenet apparatus.

3. Let $c$ be a unit speed curve with curvature $\kappa > 0$ and nonzero torsion $\tau$.
   (a) Show that if $c$ lies on a sphere with center $w$ and radius $r$ then

   \[ c - w = -\rho n - \frac{d\rho}{dt} \sigma b, \]

   where $n$ is the unit normal, $b$ the binormal, $\rho = 1/\kappa$ and $\sigma = 1/\tau$. Conclude that $r^2 = \rho^2 + \left( \frac{d\rho}{dt} \sigma \right)^2$.

   (b) Conversely, if $\rho^2 + \left( \frac{d\rho}{dt} \sigma \right)^2$ is a constant, call it $r^2$, and $\frac{d\rho}{dt} \neq 0$, show that the curve lies on a sphere of radius $r$. Hint: show that $\frac{d}{dt} \left( c + \rho n + \frac{d\rho}{dt} \sigma b \right) = \vec{0}$.

4. Compute the Frenet apparatus for the curve

   \[ c(t) = (3t - t^2, 3t^2, 3t + t^3). \]

   Caution: this is not a unit speed curve.