Review for the second midterm, Math 416h, Friday, March 30 **Prof. Eugene Lerman**

The exam will cover lectures 17–25.

1 Warm up. Give a **brief** answer to the following questions.

(a) What does it mean for a map to be multilinear?

(b) What does it mean for a multilinear map to be alternating?

(c) What's a permutation? What's the permutation group S_n?

(d) Is every permutation invertible?

(f) What's the sign of a permutation?

(g) What is the permutation representation?

(h) What's the determinant of a square matrix? What's the determinant of a linear map between two finite dimensional vector spaces?

(i) What's trace of a matrix? Of a linear map?

(j) What does it mean for two matrices to be similar? Is being similar an equivalence relation?

2 Let $S, T : V \to W$ be two linear maps, $\mathcal{B}$ a basis if V, $\mathcal{B}'$ a basis of W.
(a) Prove that $[S+T]_{\mathcal{B},\mathcal{B}} = [S]_{\mathcal{B},\mathcal{B}} + [T]_{\mathcal{B},\mathcal{B}}$.

3 Prove that $\operatorname{tr} : M_{n,n} \to \mathbb{R}$ is linear: for any two $n \times n$ matrices A, B and for any two scalars λ, μ,

$$\operatorname{tr}(\lambda A + \mu B) = \lambda \operatorname{tr}(A) + \mu \operatorname{tr}(B).$$

4 Suppose $T, S : V \to V$ are two linear maps that commute: $T \circ S = S \circ T$. Prove that if v is an eigenvector of S with eigenvalue λ then $T(v)$ is also an eigenvector of S with eigenvalue λ.

5 Suppose V is a finite dimensional vector space, $\ell_1, \ell_2, \ell_3 \in V^*$ are three linear functionals. Define $D : V \times V \times V \to \mathbb{R}$ by

$$D(v_1, v_2, v_3) := \det(\ell_i(v_j)).$$

Prove that D is an alternating trilinear map.

6 Prove that an $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. (You may use the fact that $\det(BC) = \det(B)\det(C)$ for any two matrices B, C.)

7 Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$ be a permutation in S_5.
Find σ^2 and σ^{-1}.

8 A matrix P is called a permutation matrix if its entries consists of 0's and 1's with exactly one 1 in every row and every column.
(a) Prove that $P = (e_{\sigma(1)}|e_{\sigma(2)}|\dots|e_{\sigma(n)})$ for some permutation $\sigma \in S_n$.
(b) What is the determinant of P? Hint: it must have something to do with the sign of σ.

9 Let A be an $n \times n$ matrix. How are $\det(A^4)$ and $\det((-10)A)$ are related to $\det A$?

10 Let A be a 5×5 matrix with $A = -A^T$. Prove that one of the eigenvalues of A is zero.
Hint: use determinant to show that there is $v \neq 0$ so that $Av = 0$.

11 Compute the sign of the permutation $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ either by counting inversions (see lecture 20) or by writing σ as a product of transpositions (see lecture 18).

12 Find eigenvalues of the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix}$.
Is the matrix A diagonalizable? Explain.

13 Suppose $T : V \to V$ is an **invertible** linear map and v is an eigenvector of T with eigenvalue λ.
(a) Prove that $\lambda \neq 0$.
(b) Prove that v is also an eigenvector of T^{-1}. What's the corresponding eigenvalue?

14 The goal of the exercise is to prove that for any permuation matrix P (see Problem 8) there is a natural number N so that $P^N = I$.
(a) Let σ be a permutation. Prove that there are natural numbers k, ℓ so that $k < \ell$ and $\sigma^k = \sigma^l$.
(b) Prove that if $\sigma^k = \sigma^l$ for some natural numbers k, ℓ with $k < \ell$ then $\sigma^{\ell - k} = \mathrm{id}$.
(c) Prove that if $\sigma^N = \mathrm{id}$ for some natural number N then the matrix $\rho(\sigma)^N = I$ (the map $\rho : S_n \to M_{n,n}$ was defined in lecture 20).