Review for the first midterm, Math 416h, **Prof. Eugene Lerman**
Friday, March 1

1 Warm up. Give a **brief** answer to the following questions.

(a) What does it mean for a vector space V to be finite dimensional?

(b) Is the map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2x + y - 3$ linear? Explain.

(c) Can a finite dimensional vector space have two different bases?

(d) Let V be a vector space, $B = \{v_1, \ldots, v_n\}$ a collection of n distinct vectors in V. Suppose one of these vectors is 0. Can B be linearly independent? Is it possible for B to span V?

(e) Define rank and nullity of a linear map. Define rank and nullity of an $m \times n$ matrix.

(f) Suppose A is an $n \times n$ matrix with $N(A) \neq \{0\}$. Is there a vector $b \in \mathbb{R}^n$ such that the equation $Ax = b$ have no solution? Explain.

(g) Let A be an $n \times m$ matrix and E an invertible $n \times n$ matrix. Do the matrices EA and A have the same range? the same null space? the same rank? the same nullity?

(h) Let $v_1, \ldots, v_m$ be a collection of vectors in $\mathbb{R}^n$. Form an $m \times n$ matrix A by putting v_1 as the first **row**, v_2 the second **row** and so on. Suppose the reduced row echelon form of A has no zero rows. Is the collection of vectors $\{v_1, \ldots, v_m\}$ linearly independent?

(i) Is it true that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension?

(j) Are all vector spaces finite dimensional?

2 a Find all solutions (if any) of the system

$$\begin{cases} x_1 + 2x_2 - x_3 + 2x_4 = 3 \\ 3x_1 + 7x_2 + 5x_4 = 8 \\ -x_1 + 7x_3 - 2x_4 = -1 \end{cases}.$$

b If we change the right hand side $\begin{pmatrix} 3 \\ 8 \\ -1 \end{pmatrix}$ to some other vector $b \in \mathbb{R}^3$, would the system necessarily have a solution? If so, justify; if not, find an explicit right-hand side $b \in \mathbb{R}^3$ such that the system has no solution.

3 Let $T : V \to V$ be a linear map such that $T \circ T = 0$. Prove that $R(T) \subset N(T)$. Is T necessarily the zero map? Explain.

4 Consider in the space $\mathbb{R}^5$ the vectors $v_1 = (2, -1, 1, 5, -3)^T$, $v_2 = (3, -2, 0, 0, 0)^T$ and $v_3 = (1, 1, 50, -921, 0)^T$. Can this collection of 3 vectors be completed to a basis of $\mathbb{R}^5$? If it can be, complete it. If it cannot be, explain why not.

5 A 54×54 matrix A has rank 31. Prove that if the equation $Ax = b$ has one solution, then it has infinitely many. Prove also that there is $b \in R^{54}$ so that $Ax = b$ has no solutions.

6 Consider two bases of $\mathbb{R}^2$: $\mathcal{A} = \{(1,1)^T, (0,-1)^T\}$ and $\mathcal{B} = \{(1,2)^T, (1,0)^T\}$. Suppose u is a vector in $\mathbb{R}^2$ with

$$[u]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

What is $[u]_{\mathcal{B}}$?

7 Check that the map

$$T : \mathcal{P}_n \to \mathcal{P}_{n+1}, \quad T(p) = \int_0^x p(u)\, du$$

is linear. What is the dimension of $\mathcal{P}_{n+1}/\mathrm{Range}(T)$? Explain.

8 Suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of a real vector space V, $\{b_1^*, \dots, b_n^*\}$ the dual basis of V^* Prove that for any linear map $\ell : V \to \mathbb{R}$ we have

$$\ell = \sum_{i=1}^{n} \ell(b_i)\, b_i^*.$$

9 Suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ is a basis of a real vector space V and W is another vector space. Prove that for any n vectors $w_1, \dots, w_n$ in W (not necessarily all distinct) there is a unique linear map $T : V \to W$ with $T(b_i) = w_i$, $1 \le i \le n$. [Yes, I know we proved it in class. Now it's your turn :)]

10 Let V be a finite dimensional vector space, $U \subset V$ a subspace with $0 < \dim U < \dim V$.

(a) Prove that there is a subspace W of V so that $V = U \oplus W$.

(b) Prove that the quotient V/U is isomorphic to W. Hint: consider the restriction of the projection $\pi : V \to V/U$ to W.

(c) Prove that for any linear map $T : U \to W$

$$X := \{u + T(u) \mid u \in U\}$$

is a subspace of V which is isomorphic to U.

(d) Prove that $V = X \oplus W$ and that $X \cap U = N(T)$.