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9.1

Recall A linear map $T: V \to W$ is a linear isomorphism (and then V and W are isomorphic vector spaces) iff $\exists S: W \to V$ so that $S \circ T = id_V$, $T \circ S = id_W$

Note Such S is unique: if $\exists S': W \to V$ s.t. $S' \circ T = id_V$, $T \circ S' = id_W$ Then

$$S' = S' \circ id_W = S' \circ (T \circ S) = (S' \circ T) \circ S = id_V \circ S = S.$$

Notation T^{-1} = the unique inverse of T.

We proved Lemma 8.1:

- (Thm 2.17 in text) If $T: V \to W$ is a linear bijection then its inverse is linear.

Remark Let W be a vector space, $w_1 \ldots w_m \in W$ a collection of vectors (not necessarily a basis). This collection defines a linear map

$$\Psi: \mathbb{R}^m \to W, \quad \Psi\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^m x_i w_i$$

(We proved that under the assumption that $\{w_1 \ldots w_m\}$ is a basis; but we never used the assumption)

If $\mathcal{B} = \{w_1 \ldots w_m\}$ is a basis of W then $\Psi: \mathbb{R}^m \to W$ is a bijection, hence has a linear inverse

$$\Phi: V \to \mathbb{R}^m \quad \text{with} \quad \Phi\left(\Psi\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

ie $$\Phi\left(\sum x_i w_i\right) = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad \forall \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

Notation

$$[w]_{\mathcal{B}} = \Phi(w) \quad \forall w \in W.$$

- Any linear map $T: V \to W$ is uniquely determined by its values $\{T(b_1), \ldots T(b_n)\}$ on a basis $\mathcal{B} = \{b_1 \ldots b_n\}$ of V:

(i) If $T': V \to W$ is another linear map with $T(b_i) = T'(b_i)\ \forall i$ Then $T(\sum x_i b_i) = \sum x_i T(b_i) = \sum x_i T'(b_i) = T'(\sum x_i b_i)$ for all $v = \sum x_i b_i \in V$.

(ii) Conversely, given any n vectors $w_1 \ldots w_n \in W$

There is a linear map $T: V \to W$ with $T(b_i) = w_i \quad \forall i$.
Namely

$$T(\Sigma x_i b_i) := \Sigma x_i w_i$$

Note T is the composit of two linear maps

$$\begin{tikzcd} V \arrow[rr, dashed, "T"] \arrow[dr, "\Phi = [\cdot]_B"'] & & W \\ & \mathbb{R}^n \arrow[ur, "\Psi"'] & \end{tikzcd} \qquad \Psi\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \Sigma x_i w_i$$

$T = \Psi \circ [\cdot]_B$, hence T is automatically linear.

We haven't finished proving
Corollary 8.4 A linear map $T: V \to W$ is an isomorphism $\Leftrightarrow$ for a basis $B = \{b_1 \ldots b_n\}$ of V, $\{T(b_1), \ldots T(b_n)\}$ is a basis of W.
(So in particular two finite dimensional vector spaces are isomorphic $\Leftrightarrow$ their dimensions are equal)
We prove ($\Rightarrow$).
Proof of ($\Leftarrow$): Suppose $T: V \to W$ is linear, $B = \{b_1 \ldots b_n\}$ a basis of V and $\{T(b_1), \ldots T(b_n)\}$ a basis of W.
Proof 1 $\exists$ unique linear map $S: W \to V$ with $S(T(b_i)) = b_i \; \forall i$.
Then $S \circ T = id_V$. Similarly since

$$T(S(T(b_i))) = T(b_i) \quad \forall i, \qquad T \circ S = id_W.$$

$\therefore$ T is an isomorphism.
Proof 2 Since $\{T(b_1), \ldots T(b_n)\}$ span $R(T)$,
$R(T) = W$. $\Rightarrow$ $\dim R(T) = n$ and T is onto.
On the other hand

$$\dim N(T) = \dim V - \dim R(T) = n - n = 0.$$

$\Rightarrow$ T is 1-1.
By 8.1, T is an isomorphism $\square$

Linear maps and matrices (sections 2.2 – 2.4)

Recall An $m \times n$ matrix (a_{ij}) defines a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{pmatrix}$$

Lemma 9.1 Any linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ uniquely determines a matrix (t_{ij}) so that

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (t_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{for all} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Proof (1) Since T is linear, it is uniquely determined by its values on the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ $\left(e_i = (0 \dots \overset{i\text{th}}{1}, \dots 0)^T \right)$

(2) $T(e_j) \in \mathbb{R}^m$ so has to be of the form

$$T(e_j) = \begin{pmatrix} t_{1j} \\ t_{2j} \\ \vdots \\ t_{mj} \end{pmatrix} \quad \text{for some } t_{1i}, t_{2i}, \dots t_{mi} \in \mathbb{R}$$

(3)
$$\begin{pmatrix} t_{11} & \dots & t_{1j} & \dots & t_{1n} \\ \vdots & & t_{2j} & & \vdots \\ \vdots & & \vdots & & \vdots \\ t_{m1} & \dots & t_{mj} & & t_{mn} \end{pmatrix}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ slot} = \begin{pmatrix} t_{1j} \\ \vdots \\ \vdots \\ t_{mj} \end{pmatrix}$$

(4) Conclusion
$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (t_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{for } \underline{\text{all}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Notation Suppose $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear. The ij^{th} entry of the corresponding matrix is denoted by A_{ij} or by a_{ij}.

Thus $A = (A_{ij}) = (a_{ij})$

Matrix multiplication

Lemma 9.2 Let $A: \mathbb{R}^n \to \mathbb{R}^m$, $B: \mathbb{R}^m \to \mathbb{R}^\ell$ be two linear maps. Then

$$(B \circ A)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}.$$

Proof On the one hand

$$(B\circ A)(e_j) = \begin{pmatrix} (B\circ A)_{1j} \\ \vdots \\ (BA)_{ij} \\ \vdots \\ (B\circ A)_{\ell j} \end{pmatrix} \leftarrow i^{th} \text{ slot}.$$

On the other hand

$$(B\circ A)(e_j) = B(A(e_j)) = B\begin{pmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{pmatrix} = B\left(\sum_{k=1}^{m} A_{kj} e_k\right) =$$

$$= \sum_{k=1}^{m} A_{kj} B(e_k) = \sum_{k=1}^{m} A_{kj} \begin{pmatrix} B_{1k} \\ \vdots \\ B_{\ell k} \end{pmatrix} = \begin{pmatrix} \sum A_{kj} B_{1k} \\ \vdots \\ \sum_k A_{kj} B_{ik} \\ \vdots \\ \sum A_{kj} B_{\ell k} \end{pmatrix} \leftarrow i^{th} \text{ slot}$$

$$\therefore \quad (BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$$

Definition Given an $m\times n$ matrix (a_{kj}) and an $\ell \times m$ matrix (b_{ik}) their **product** is the $\ell \times n$ matrix (c_{ij}) with

$$c_{ij} := \sum_{k=1}^{m} b_{i\underline{\underline{k}}}\, a_{\underline{\underline{k}}j}$$

Ex
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0\cdot1 + 1\cdot2 + 0\cdot3 & 0\cdot4 + 1\cdot5 + 0\cdot6 \\ 1\cdot1 + 0\cdot2 + 0\cdot5 & 1\cdot4 + 0\cdot5 + 0\cdot6 \\ 0\cdot1 + 0\cdot2 + 1\cdot3 & 0\cdot4 + 0\cdot5 + 1\cdot6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 1 & 4 \\ 3 & 6 \end{pmatrix}.$$