Last time Rank/nullity ("dimension") theorem:

- if $\dim V < \infty$, $T: V \to W$ is linear then
$$\dim V - \dim N(T) = \dim R(T).$$
- $T: V \to W$ is injective $\Leftrightarrow N(T) = \{\vec{0}\} \Leftrightarrow \dim N(T) = 0.$
- Suppose $\dim V = \dim W < \infty$ and $T: V \to W$ is linear. Then
  T is injective $\Longleftrightarrow$ T is surjective $\Longleftrightarrow$ T is a bijection.

Hence either $Tx = b$ has a unique solution $\forall b \in W$

or $\exists\, b$ s.t. $Tx = b$ has no solutions and
$Tx = \vec{0}$ has infinitely many solutions

<insert1 goes here>

"Recall" For any vector space V we have a map $id_V \equiv I_V : V \to V$
$$id_V(x) = x \quad \forall x \in V.$$
id_V is linear (check that!)

Definition (compare with Def on p99 of text). A linear map $T: V \to W$ is invertible (or is a linear isomorphism) if there is a linear map $S: W \to V$ such that
$$T \circ S = id_W, \quad S \circ T = id_V.$$

Lemma 8.1 A linear map $T: V \to W$ is an isomorphism $\Longleftrightarrow$ T is a bijection (ie 1-1 and onto).

Proof ($\Rightarrow$). Suppose $T: V \to W$ is an isomorphism. Then $\exists$ linear map $S: W \to V$ s.t. $S \circ T = id_V$, $T \circ S = id_V$.
If $T(v_1) = T(v_2)$ then $v_1 = S(T(v_1)) = S(T(v_2)) = v_2 \Rightarrow T$ is 1-1
$\forall w \in W$. $T(S(w)) = w. \Rightarrow T$ is onto.

($\Leftarrow$) Suppose $T: V \to W$ is a linear bijection. Then there is a unique map $S: W \to V$ st $S \circ T = id_V$, $T \circ S = idW$

We need to check that S is linear!

$\forall\, w_1, w_2 \in W$, $\forall\, \lambda_1, \lambda_2 \in \mathbb{R}$ $\quad T \circ S = id_W$

$$T(S(\lambda_1 w_1 + \lambda_2 w_2)) = \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 T(S(w_1)) + \lambda_2 T(S(w_2))$$

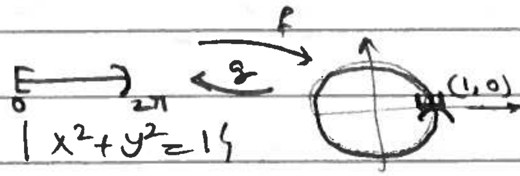
$$= T(\lambda_1 S(w_1) + \lambda_2 S(w_2)) \quad \text{since } T \text{ is linear.}$$

$$\Rightarrow \quad S(\lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 S(w_1) + \lambda_2 S(w_2) \quad \text{since } T \text{ is } 1\text{-}1.$$

□

Rant The book defines a linear map $T: V \to W$ to be invertible (to be an isomorphism) if T has an inverse as a function.
This is morally wrong because such approach fails in other contexts:

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is differentiable and $g(y) = y^{1/3}$ is an inverse of f, but g is not differentiable at 0.



Ex $f: [0, 2\pi) \longrightarrow S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$
$f(\theta) = (\cos\theta, \sin\theta)$ is continuous bijection.
So f has an inverse $g: S^1 \longrightarrow [0, 2\pi)$. This inverse is not continuous at $(1,0) \in S^1$.

<insert 1>.

Ex Suppose $T: V \to W$ is linear, $\dim V, \dim W < \infty$ and $\dim W < \dim V$.
Then T cannot be $1\text{-}1$:

Proof $\dim V > \dim W \geq \dim R(T) = \dim V - \dim N(T)$.

$$\Rightarrow \quad 0 > -\dim N(T) \Rightarrow \dim N(T) > 0$$

$$\Rightarrow \quad N(T) \neq \{0\}$$

$$\Rightarrow \quad T \text{ is not } 1\text{-}1.$$

□

An "application" of 8.1 : bases defined coordinates.

Lemma 8.2 A basis $\mathcal{B}=\{b_1,\dots,b_n\}$ of a vector space V defines a linear isomorphism $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ with $\left[\sum_{i=1}^n x_i b_i\right]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Proof. Since $\mathcal{B}=\{b_1 \dots b_n\}$ is a basis, for any $v \in V$ $\exists$ unique $x_1, \dots x_n \in \mathbb{R}$ such that $v = \sum_{i=1}^n x_i b_i$.

$\Rightarrow$ The map $T: \mathbb{R}^n \to V$, $T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n x_i b_i$ is a bijection, hence has an inverse $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ so that $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left[T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right]_{\mathcal{B}} = \left[\sum x_i b_i\right]_{\mathcal{B}}$.

We need to check that $[\cdot]_{\mathcal{B}}$ is a linear isomorphism.
By 8.1, it's enough to check that T is linear.
Now $\forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ $\forall \lambda, \mu \in \mathbb{R}$

$$T\left(\lambda\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \mu\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right) = T\begin{pmatrix} \lambda x_1 + \mu y_1 \\ \vdots \\ \lambda x_n + \mu y_n \end{pmatrix} = \sum_{i=1}^n (\lambda x_i + \mu y_i) b_i$$

$$= \lambda\left(\sum_{i=1}^n x_i b_i\right) + \mu\left(\sum_{i=1}^n y_i b_i\right) = \lambda T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \mu T\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad \square$$

Theorem 8.3 (important!)
A linear map is uniquely determined by what it does to a basis.
That is, given a basis $\mathcal{B}=\{b_1 \dots b_n\}$ of a vector space V and a set of vectors $w_1, \dots w_n$ in a vector space W (not nec. distinct), there is a unique linear map $T: V \to W$ so that $T(b_1)=w_1, \dots T(b_n)=w_n$.

Proof Given $v \in V$ there are unique $x_1 \dots x_n \in \mathbb{R}$ s.t. $v = \sum x_i b_i$.
We define

$$T(v) := \sum x_i w_i .$$

Then $T(b_1) = T(1\cdot b_1 + 0 b_2 + \dots + 0 b_n) = 1 w_1 + 0 w_2 + \dots + 0 w_n = w_1$

$T(b_2) = T(0\cdot b_1 + 1 b_2 + 0 b_3 + \dots + 0 b_n) = 0\cdot w_1 + 1\cdot w_2 + 0 w_3 + \dots + 0 w_n = w_2$

etc. $\Rightarrow T(b_i) = w_i \ \forall i$.

It's not hard to check that T is linear (see proof of 8.2)

Moreover, suppose $S: V \to W$ is linear and $S(b_1) = w_1, \ldots S(b_n) = w_n$.

Then $\forall x_1 \ldots x_n \in \mathbb{R}$

$$S(\Sigma x_i b_i) = \Sigma x_i S(b_i) = \Sigma x_i w_i = \Sigma x_i T(b_i) = T(\Sigma x_i b_i)$$

Since $\forall v \in V \ \exists x_1 \ldots x_n \in \mathbb{R}$ with $v = \Sigma x_i b_i$,

$$S(v) = T(v) \ \forall v \in V. \Rightarrow S = T$$ ☐

Corollary 8.4 A linear map $T: V \to W$ is a linear isomorphism $\Leftrightarrow$ for a basis $\mathcal{B} = \{b_1 \ldots b_n\}$ of V, $\{T(b_1), \ldots T(b_n)\}$ is a basis of W.

Proof ($\Rightarrow$) Suppose $T: V \to W$ is an isomorphism and $\mathcal{B} = \{b_1 \ldots b_n\}$ a basis of V. Since T is an iso, it has an inverse $S: W \to V$.

Since $\mathcal{B}$ is a basis of V, $\exists x_1 \ldots x_n \in \mathbb{R}$ s.t. $S(w) = \Sigma x_i b_i$.

$\Rightarrow w = T(S(w)) = T(\Sigma x_i b_i) = \Sigma x_i T(b_i)$. $\Rightarrow \{T(b_1), \ldots T(b_n)\}$ spans W.

If $0 = \Sigma c_i T(b_i)$, Then $0 = S(0) = S(\Sigma c_i T(b_i)) = \Sigma c_i ST(b_i) = \Sigma c_i b_i$.

$\rightarrow c_1 = c_2 \cdots = c_n = 0$ since $\{b_1 \ldots b_n\}$ is a basis.

$\Rightarrow \{T(b_1), \ldots T(b_n)\}$ is linearly independent.

($\Leftarrow$)

Suppose $\{T(b_1), \ldots T(b_n)\}$ is a basis of W. By 8.3 $\exists$ unique linear map $S: W \to V$ s.t. $S(T(b_i)) = b_i \ \forall i$.

We argue That $S \circ T = id_V$ and $T \circ S = id_W$.

Since $\forall i$ $S \circ T(b_i) = S(T(b_i)) = b_i = id_V(b_i)$, $S \circ T = id_V$.

Since $\forall i$, $(T \circ S)(T(b_i)) = T(S(T(b_i))) = T(b_i) = id_W(T(b_i))$

$$T \circ S = id_W.$$ ☐

8.4 "explains" where $[\cdot]_{\mathcal{B}}: V \to \mathbb{R}^n$ comes from:

$[\cdot]_{\mathcal{B}}$ is the unique iso that takes the basis $\mathcal{B}$ of V the standard basis $\{e_1 \ldots e_n\}$ of $\mathbb{R}^n$.