02/03/2019 VRC 7.1

Last time • A basis of a subspace $W$ of a finite dim vector space $V$ can be extended to a basis of $V$

- The sum $U+W$ of subspaces of a v. space $V$.
- elementary properties of linear maps, null space / kernel range, nullity and rank.

Recall $N(T: V \to W) = \{v \in V \mid T(v) = 0\}$, $R(T) = \{T(v) \mid v \in V\} \equiv$ image of $T$. (see p70

Stated but didn't have time to prove: ( Text calls it dimension theorem )

Rank/nullity Theorem: $T: V \to W$ linear, $V$ finite dimensional

$\Rightarrow$ $\dim V - \dim N(T) = \dim R(T)$.

Proof Since $N(T) := \{v \in V \mid T(v) = \vec{0}\}$ is a subspace of a finite dimensional vector space, it has a finite basis. Call it $\{y_1 \ldots y_m\}$. We can extend it to a basis $\{y_1 \ldots y_m, z_1 \ldots z_k\}$ of $V$ (so that $\dim V = m+k$).

Claim $\{T(z_1), \ldots T(z_k)\}$ is a basis of $R(T) = T(V)$.

Note that claim $\Rightarrow$ $\dim R(T) = k = m+k-k = \dim V - \dim N(T)$

Proof of claim

(1) $\forall\, w \in R(T)$ $\exists\, v \in V$ s.t. $w = T(v)$.

Since $\{y_1 \ldots y_m, z_1 \ldots z_k\}$ is a basis of $V$ $\exists\, d_1, \ldots d_m, \lambda_1 \ldots \lambda_k \in \mathbb{R}$ so that $v = \sum_{i=1}^{m} d_i y_i + \sum_{j=1}^{k} \lambda_j z_j$

Since $y_1 \ldots y_k \in N(T)$; $T(y_1) = T(y_2) \ldots = T(y_m) = \vec{0}$

$\Rightarrow w = T(v) = \sum d_i T(y_i) + \sum \lambda_j T(z_j) = \sum \lambda_j T(z_j)$.

$\therefore$ $\{T(z_1), \ldots, T(z_k)\}$ spans $R(T)$.

Suppose $\exists\, c_1 \ldots c_k \in \mathbb{R}$ s.t.

$$c_1 T(z_1) + \ldots + c_k T(z_k) = \vec{0}$$

Then $\vec{0} = T(c_1 z_1 + \ldots + c_k z_k)$.

$\Rightarrow$ $c_1 z_1 + \ldots + c_k z_k \in N(T)$

$\Rightarrow$ $c_1 z_1 + \ldots + c_k z_k = d_1 y_1 + \ldots + d_m y_m$ for some $d_1 \ldots d_m \in \mathbb{R}$. But $\{y_1 \ldots y_m, z_1 \ldots z_m\}$ is a basis

$\Rightarrow c_1 = c_2 \cdots = c_k = d_1 = \cdots = d_m = 0$

$\therefore \{T(z_1), \ldots T(z_k)\}$ is linearly independent.

$\square$

**Aside.** What does $\dim N(T)$ measure?

**Lemma 7.1** Let $T: V \to W$ be a linear map. For any $v_1, v_2 \in V$

$$T(v_1) = T(v_2) \iff v_1 - v_2 \in N(T)$$

Consequently $T$ is 1-1 $\iff N(T) = \{\vec{0}\} \iff \dim N(T) = 0$.

**Proof** • $T(v_1) = T(v_2) \iff \vec{0} = T(v_1) - T(v_2) = T(v_1 - v_2)$

$\iff v_1 - v_2 \in N(T)$.

• $T$ is 1-1 $\iff \forall v_1, v_2 \in V \quad T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

$\iff \forall v_1, v_2 \in V \quad T(v_1 - v_2) = \vec{0} \Rightarrow v_1 - v_2 = \vec{0}$

$\iff \forall x \in V \quad T(x) = \vec{0} \Rightarrow x = \vec{0}$

$\iff N(T) = \{\vec{0}\}$. $\square$

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**Lemma 7.2** Suppose $T: V \to W$ is linear, $\dim V, \dim W < \infty$ and $\dim V = \dim W$. Then

$$T \text{ is 1-1} \iff T \text{ is onto.}$$

**Proof** $T$ is onto $\iff R(T) = W \iff \dim W = \dim R(T)$.

$\dim R(T) = \dim V - \dim N(T)$ by rank/nullity.

Therefore,

$T$ is onto $\iff \dim V = \dim W = \dim R(T) = \dim V - \dim N(T)$

$\iff \dim N(T) = 0$

$\iff N(T) = \{\vec{0}\} \iff T$ is 1-1. $\square$

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**Corollary 7.3** Suppose $T: V \to W$ is linear, $\dim V = \dim W < \infty$ as in 7.2. Then either

• The equation $Tx = b$ has exactly one solution $\forall b \in W$

Or $\exists b \in W$ st $Tx = b$ has no solutions
<u>and</u> $Tx = \vec{0}$ has infinitely many solutions

<u>Proof</u> • $Tx = \vec{0}$ has exactly one solution $\overset{(7.1)}{\Leftrightarrow}$ $N(T) = \{\vec{0}\}$
$\overset{(7.1)}{\Leftrightarrow}$ $T$ is 1-1 $\overset{(7.2)}{\Leftrightarrow}$ $T$ is onto
$\Leftrightarrow$ $\forall\, b \in W$, $Tx = b$ has exactly one solution.

• $Tx = \vec{0}$ has more than one solution
$\Leftrightarrow$ $T$ is <u>not</u> 1-1, $\Leftrightarrow$ $N(T) \neq \{0\}$
$\Leftrightarrow$ $\dim N(T) > 0$ (and $Tx = \vec{0}$ has infinitely many solutions)
$\Leftrightarrow$ $\dim R(T) = \dim V - \dim N(T) < \dim V = \dim W$
$\Leftrightarrow$ $R(T) \neq W$ $\Leftrightarrow$ $T$ is <u>not</u> onto
$\Leftrightarrow$ $\exists b \in W$ st $Tx = b$ has <u>no</u> solutions.

□.

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<u>Recall</u> $A$ $m \times n$ matrix $A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
defines a linear map

$$A : \mathbb{R}^n \to \mathbb{R}^m, \quad A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

The <u>nullity of the matrix $A$</u> is, by definition

$$\dim\{x \in \mathbb{R}^n \mid Ax = \vec{0}\} = \dim(N(A))$$

The nullity of the <u>linear map</u> $A$.

The <u>rank of the matrix $A$</u> is, by definition,

$$\dim\left(\operatorname{span}\left\{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}\right\}\right)$$

$$= \dim(\operatorname{span}\{\text{columns of } A\})$$

Now $w \in \operatorname{span}\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \ldots \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}\right) \Leftrightarrow$

$$w = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

for some $x_1, \dots, x_n \in \mathbb{R}$

$\Leftrightarrow \quad w = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\Leftrightarrow \quad w \in R(A).$

Hence rank of a matrix $A$ = rank of the linear map $A: \mathbb{R}^n \to \mathbb{R}^m$.

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Lemma 7.4 Let $T: V \to W$ be a linear map, suppose $\dim V = n$ and $\{v_1, \dots, v_n\}$ is a basis of $V$. Then $\{T(v_1), \dots, T(v_n)\}$ spans $R(T)$

**Proof** Since $\{T(v_1), \dots, T(v_n)\} \subseteq R(T)$ and $R(T)$ is a subspace $\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$, $\lambda_1 T(v_1) + \dots + \lambda_n T(v_n) \in R(T)$

$\Rightarrow \quad \operatorname{span}\{T(v_1), \dots, T(v_n)\} \subseteq R(T).$

Conversely, given $w \in R(T)$ $\exists\, v \in V$ st $w = T(v)$

Since $\{v_1, \dots, v_n\}$ is a basis $\exists\, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ st $v = \sum \lambda_i v_i$

$\rightarrow \quad w = T(v) = T(\sum \lambda_i v_i) = \sum \lambda_i T(v_i) \in \operatorname{span}\{T(v_1), \dots, T(v_n)\}$

$\Rightarrow \quad R(T) \subseteq \operatorname{span}\{T(v_1), \dots, T(v_n)\}.$

**Remark** Given an $m \times n$ matrix $A = (a_{ij})$ the $j^{th}$ column of $A$ is $Ae_j$ where $e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow j^{th}$ slot.

Check
$$Ae_j = \begin{pmatrix} a_{11}0 + \dots + a_{1j-1}0 + a_{1j}1 + a_{1j+1}0 - \dots + a_{1n}0 \\ a_{21}0 + \dots + a_{2j}1 + \dots + a_{2n}0 \\ \vdots \\ a_{m1}0 + \dots + a_{mj}1 + \dots + a_{mn}0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$