Last time: V = a vector space

- If $S \subseteq V$ is linearly independent, $v \in V$, $v \notin \operatorname{span} S$ then $S \cup \{v\}$ is linearly independent.

Now $\boxed{\text{assume } V \text{ is finite dimensional}}$

- If $S \subseteq V$, $V = \operatorname{span}(S)$ and $|S| = \dim V$, then S is a basis of V.
- If $S \subseteq V$, S is linearly independent and $|S| = \dim V$ then S is a basis of V.
- (5.2) If $W \subseteq V$ is a subspace then
  (i) W is finite dimensional and $\dim W \leq \dim V$
  (ii) if $\dim W = \dim V$, $W = V$.

Corollary 5.3 (corollary on p51 of text). Let W be a subspace of a finite dimensional vector space V. Then any basis of W can be extended to a basis of V.

Proof Let S be a basis of W (by 5.2 we know that W has a finite basis). Then $S \subseteq V$ is linearly independent. Hence by 5.1(c) $\exists$ $S' \subseteq V$, s.t. S' is a basis of V and $S \subseteq S'$!

$\square$

Ex $V = \mathbb{R}^3$ $W = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

$(-1,1,0)$, $(-1,0,1) \in W$. Moreover $\forall$ $(x_1, x_2, x_3)^T \in W$

$$x_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-x_2 - x_3\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad \text{since } x_1 = -x_2 - x_3.$$

$\Rightarrow \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ spans W.

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = c_1\begin{pmatrix}-1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\0\\1\end{pmatrix} \Rightarrow \begin{pmatrix}-c_1 - c_2\\c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0.$$

$\Rightarrow S = \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ is a basis of W.

$\begin{pmatrix}+1\\0\\0\end{pmatrix} \notin W.$ $\Rightarrow$ $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ is lin independent in $\mathbb{R}^3$.

$\Rightarrow$ it's a basis of $\mathbb{R}^3$ extending the basis
$S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ of W.

Sums of subspaces

Let V be a vector space, $U, W \subseteq V$ two subspaces.
Their <u>sum</u> is the set

$$U+W = \{ u+w \mid u \in U,\ w \in W \}.$$

<u>Note</u> $\forall\ u_1, u_2 \in U,\ w_1, w_2 \in W,\ \lambda_1, \lambda_2 \in \mathbb{R}$

$$\lambda_1 (u_1 + w_1) + \lambda_2 (u_2 + w_2) = (\lambda_1 u_1 + \lambda_2 u_2) + (\lambda_1 w_1 + \lambda_2 w_2) \in U+W$$

$\Rightarrow$ $U+W$ is a <u>subspace</u> of V.

<u>Ex</u> $V = \mathbb{R}^4$ $\quad U = \{ (x_1, x_2, 0, 0)^T \in \mathbb{R}^4 \mid x_1, x_2 \in \mathbb{R} \}$

$W = \{ (0, y_1, y_2, 0)^T \in \mathbb{R}^4 \mid y_1, y_2 \in \mathbb{R} \}$

$$U+W = \{ (x_1,\ x_2 + y_1,\ y_2,\ 0)^T \mid y_1, y_2, x_1, x_2 \in \mathbb{R} \}$$
$$= \{ (z_1, z_2, z_3, 0)^T \in \mathbb{R}^4 \mid z_1, z_2, z_3 \in \mathbb{R} \}$$

<u>Ex</u> V any vector space, $U \subseteq V$ any subspace

<u>Claim</u> $U + U = U$

<u>Reason</u> $\forall\ u_1, u_2 \in U$, $u_1 + u_2 \in U$ $\Rightarrow$ $U + U \subseteq U$.

On the other hand $\forall\ u \in U$ $\quad u = u + \vec{0} \in U+U$

$\Rightarrow$ $U \subseteq U + U$. $\square$

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<u>Exercise</u> Suppose V is a finite dimensional vector space, $U, W \subseteq V$.
Then $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

<u>Hint:</u> Choose a basis $\{x_1 \ldots x_k\}$ of $U \cap W$

Extend it to a basis $\{x_1 \ldots x_k, y_1 \ldots y_e\}$ of U

$\{x_1 \ldots x_k, z_1 \ldots z_n\}$ of W

Check that $\{x_1 \ldots x_m, y_1 \ldots y_l, z_1 \ldots z_n\}$ is a basis of $U+W$.

Linear maps. (ch 2 in text)

Recall A map $T: V \to W$ between two vector spaces is **linear** if $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $v_1, v_2 \in V$

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2).$$

Definition The **range** of a linear map $T: V \to W$ is the image of T:

$$R(T) := \{T(v) \mid v \in V\}$$

The **null space** or **kernel** of T is

$$N(T) \equiv \ker T = \{v \in V \mid T(v) = \vec{0}\}.$$

Remark Suppose $T: V \to W$ is linear. Then

1) $T(\vec{0}_V) = \vec{0}_W$
2) $T(-v) = -(T(v)) \quad \forall v \in V$
3) $\forall n > 1$, $\forall \lambda_1 \ldots \lambda_n \in \mathbb{R}$ $\forall v_1 \ldots v_n \in V$
$$T\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i T(v_i).$$

Proof 1) $T(\vec{0}_V) = T(0 \cdot \vec{0}_V) = 0 \cdot T(\vec{0}_V) = \vec{0}_W$

2) $T(-v) + T(v) = T(-v+v) = T(\vec{0}) = \vec{0}$
$\Rightarrow T(-v) = -(T(v))$.

3) induction on n.

Lemma 6.1 (Thm 2.1 in text) Let $T: V \to W$ be a linear map. Then

1) $N(T) = \{v \in V \mid T(v) = \vec{0}\}$ is a subspace of V
2) $R(T) = \{T(v) \in W \mid v \in V\}$ is a subspace of W.

Proof 1) $\forall\ v_1, v_2 \in N(T)\quad \forall\ \lambda_1, \lambda_2 \in \mathbb{R}$

$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2) = \lambda_1 \vec{0} + \lambda_2 \vec{0} = \vec{0}$

$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in N(T)$. $\Rightarrow N(T)$ is a subspace of V

2) Suppose $w_1, w_2 \in R(T)$. Then $\exists\, v_1, v_2 \in V$ s.t. $w_i = T(v_i)$, $i=1,2$

$\Rightarrow \forall\ \lambda_1, \lambda_2 \in \mathbb{R}$

$$\lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 T(v_1) + \lambda_2 T(v_2) = T(\lambda_1 v_1 + \lambda_2 v_2) \in R(T)$$

$\Rightarrow R(T)$ is a subspace of W. $\square$

Definition The rank of $T: V \to W$ is $\operatorname{rank}(T) = \dim R(T)$

The nullity of $T: V \to W$ is $\operatorname{null}(T) = \dim N(T)$.

Rank/Nullity Theorem (Thm 2.3 in text)

Let V be a finite dimensional vector space, $T: V \to W$ linear.

Then $\boxed{\dim V - \dim N(T) = \dim R(T)}$

Note Since V is finite dimensional, it has a basis $\{x_1, \ldots, x_n\}$.

$\Rightarrow \forall\ v \in V$, $\exists$ (unique) $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. $v = \sum \alpha_i x_i$

$\Rightarrow T(v) = T(\sum \alpha_i x_i) = \sum \alpha_i T(x_i)$.

$\Rightarrow \{T(x_1), \ldots, T(x_n)\}$ spans $R(T)$

$\Rightarrow \dim R(T)$ makes sense and $\dim R(T) \le \dim V$.

Idea of one proof of rank/nullity theorem:

Pick a basis $\{y_1, \ldots, y_m\}$ of $N(T)$, extend it to a basis $\{y_1, \ldots, y_m, z_1, \ldots, z_k\}$ of V. $T(y_i) = \vec{0}\quad \forall i$

$\Rightarrow R(T) = \operatorname{span}\{T(z_1), \ldots, T(z_k)\}$

A computation shows: $\{T(z_1), \ldots, T(z_k)\}$ is linearly independent, hence a basis of $R(T)$. $\Rightarrow$

$\dim R(T) = k = k + m - m = \dim V - \dim(N(T))$. $\square$