Last time:

4.1: If a vector space V is generated by a finite set S then a subset of S is a basis of V (p44 Thm 1.9 in Friedberg et al.)

4.2: Suppose V is a vector space $\{x_1, \dots x_n\} \subseteq V$ spans V and $\{y_1, \dots, y_m\} \subseteq V$ is linearly independent then $m \leq n$. (p 45 Thm 1.10

Note 5.0 We proved, in fact, that $\exists\, i_1, \dots i_{n-m} \in \{1, \dots n\}$ such that $\{y_1, \dots y_m, x_{i_1}, \dots x_{i_{n-m}}\}$ spans V.

4.3 Suppose $\{x_1, \dots x_n\}$, $\{y_1, \dots y_m\}$ are two bases of a vector space V. Then $n = m$.

Definition A vector space V is finite dimensional if it has a finite basis. The dimension of V is the number of vectors in the bases.

Note By 4.3 the dimension of a finite dimensional vector space is well-defined.

Notation: $\dim V$ = dimension of a vector space V.
$|X|$ = # of elements in a set X.

Remark If a vector space V is spanned by a finite set then V is finite dimensional by 4.1.

Lemma 5.1 (equivalent to Cor 2 on p 47)
Let V be a vector space of dimension $n \geq 0$ ($n \in \mathbb{Z}$)

(a) If a set $S \subseteq V$ spans (generates) V, then $|S| \geq n = \dim V$
If $|S| = n$, then S is a bases.

(b) If $S \subseteq V$ is linearly independent and $|S| = n = \dim V$ then S is a basis of V.

(c) If $S \subseteq V$ is linearly independent then $\exists\, S' \subseteq V$, $S \subseteq S'$

so that S' is a basis of V.

Proof (a) By 4.1, a subset of S is a basis for V. By 4.3 this subset has $n = \dim V$ elements. $\Rightarrow$ S has n or more elements. If S has $n = \dim V$ elements and $S' \subsetneq S$ then $|S'| < n$ so cannot be a basis. But some subset of S is a basis. So it has to be S itself.

(b) We want to argue that $\operatorname{span} S = V$.
Suppose not. Then $\exists v \in V$ s.t. $v \notin \operatorname{span} S$. (in particular $v \notin S$)
Claim: $S \cup \{v\}$ is linearly independent.
Proof of Claim: if $S \cup \{v\}$ is lineary dependent, $\exists$ $y_1 \ldots y_m \in S$ and $\alpha_1 \ldots \alpha_m, \alpha_{m+1} \in \mathbb{R}$ not all zero st

$$\alpha_1 y_1 + \ldots + \alpha_m y_m + \alpha_{m+1} v = \vec{0}.$$

α_{m+1} cannot be zero, for then $\alpha_1 y_1 + \ldots + \alpha_m y_m = \vec{0}$ which contradicts linear independence of S.
Since $\alpha_{m+1} \neq 0$, $\quad v = \left(-\frac{1}{\alpha_{m+1}}\right)(\alpha_1 y_1 + \ldots + \alpha_m y_m) \in \operatorname{Span} S$
contradiction again.
Since $S \cup \{v\}$ is linearly independent $n \geq |S \cup \{v\}|$.
Since $v \notin S$, $|S \cup \{v\}| = n+1$. Contradiction.

(c) Suppose $S = \{y_1 \ldots y_m\}$ is linearly independent and $\{x_1 \ldots x_n\}$ is a basis of V. By Note 5.0 $\exists$ $x_{i_1} \ldots x_{i_{n-m}}$ s.t.

$$\{y_1 \ldots y_m, x_{i_1} \ldots x_{i_{n-m}}\} \text{ spans } V.$$

The set $\{y_1 \ldots y_m, x_{i_1} \ldots x_{i_{n-m}}\}$ has at most n elements.
Since it spans V it has to have at least n elements.
$\Rightarrow S' = \{y_1 \ldots y_m, x_{i_1} \ldots x_{i_{n-m}}\}$ has exactly n elements
By (a) the set S' is a basis. $\square$

Ex Let $V = \mathbb{R}^2$ $\{(1,1)^T\} = S$ is linearly independent since $(1,1)^T \neq (0,0)^T$.

$\{e_1 = (1,0)^T, e_2 = (0,1)^T\}$ is a basis of $\mathbb{R}^2$.

$e_1, e_2 \notin \text{span}\{(1,1)^T\}$.

$\Rightarrow$ $\{(1,1)^T, e_1\}$ and $\{(1,1)^T, e_2\}$ are both linearly independent sets with $2 = \dim \mathbb{R}^2$ elements.

$\Rightarrow$ $S' = \{e_1, (1,1)^T\}$ and $S'' = \{(1,1)^T, e_2\}$ are bases of $\mathbb{R}^2$.

$\square$

Theorem 5.2 (Thm 1.11 on p 50) Suppose W is a subspace of a finite dimensional vector space V. Then

(1) W is finite dimensional

(2) $\dim W \leq \dim V$

(3) if $\dim W = \dim V$, $W = V$.

Remark The proof uses repeatedly: if $S \subseteq W$ is linearly independent and $\exists w \in W$ s.t. $w \notin \text{span}\, S$ Then $S \cup \{w\}$ is linearly independent.

Proof of 5.2 If $W = \{\vec{0}\}$, $\dim W = 0 \leq \dim V$.

Suppose $W \neq \{\vec{0}\}$. Then $\exists x_1 \in W$ s.t. $x_1 \neq \vec{0}$. If $W = \text{span}\{x_1\}$ and $\dim W = 1$. If $W \neq \text{span}\{x_1\}$, $\exists x_2 \in W$ s.t. $x_2 \notin \text{span}\{x_1\}$.

Then $\{x_1, x_2\}$ is lin. independent. If $W = \text{span}\{x_1, x_2\}$, we're done. Otherwise keep choosing $x_3, x_4, \ldots x_k$ s.t. $\{x_1, \ldots x_k\}$ is lin. independent.

Note that by 4.2 $k \leq \dim V$, so the process has to stop after at most $\dim V$ steps.

The process stops when $W = \text{span}\{x_1, \ldots, x_k\}$.

$\Rightarrow \dim W = k \leq \dim V$. If $k = \dim V$, $\{x_1 \ldots x_k\}$ is

a basis of V by 5.1 (b). In this case
$W = \text{span}\{x_1, \dots, x_k\} = V$. $\square$

Corollary 5.3 (Corollary on p 51) Let W be a subspaces of a finite dimensional vector space V. Then any basis of W can be extended to a basis of V.

<u>Proof</u> Note first that by 5.2, W has a finite basis S.
Then S is linearly independent as a subset of W
$\Rightarrow S$ is linearly independent as a subset of V.
By 5.1 (c) $\exists\, S' \subseteq V$ s.t (1) $S \subseteq S'$ and (2) S' is a basis of V.

<u>Ex</u> Let $V = \mathbb{R}^3$, $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_1 + x_2 + x_3 = 0 \right\}$

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in W$. Moreover, for any $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W$

$$x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{since } x_1 = -x_2 - x_3.$$

$\Rightarrow \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans W

$$c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = 0$$

$\Rightarrow S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of W.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin W = \text{span}\, S$. $\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$
is lin. independent in $\mathbb{R}^3$. $\Rightarrow$ it's a basis of $\mathbb{R}^3$ extending the basis of W.