Recall Let V be a vector space, $S \subseteq V$ a subset.

$$\operatorname{span}(S) := \{ \alpha_1 v_1 + \dots + \alpha_n v_n \mid n > 0,\ v_1 \dots v_n \in S,\ \alpha_1, \dots, \alpha_n \in \mathbb{R} \}$$

(by convention $\operatorname{span} \emptyset = \{\vec{0}\}$)

Facts: 1) $\operatorname{span}(S)$ is a subspace of V

2) $\operatorname{span}(S)$ = the intersection of all subspaces containing S

(we haven't proved them in class...)

Definition A subset S of a vector space V is linearly independent if for any $m > 0$, any distinct vectors $v_1, \dots v_m \in S$

$$\alpha_1 v_1 + \dots + \alpha_m v_m = \vec{0} \Rightarrow \alpha_1 = 0,\ \alpha_2 = 0, \dots \alpha_m = 0 \quad \forall \alpha_1 \dots \alpha_m \in \mathbb{R}$$

A subset S of V is linearly dependent if for some $n > 0$ there are n vectors $v_1, \dots v_n \in S$ and $\alpha_1, \dots \alpha_n \in \mathbb{R}$ not all zero s.t.

$$\alpha_1 v_1 + \dots + \alpha_n v_n = \vec{0}.$$

Example $\{x^i\}_{i=0}^{\infty} \subseteq \mathbb{R}[x]$ is linearly independent: given $m > 0$ for any $i_1, \dots i_m \in \mathbb{N}$ (all distinct) and any $a_1, \dots a_m \in \mathbb{R}$

if $\quad a_1 x^{i_1} + \dots + a_m x^{i_m} = 0 \leftarrow$ zero polynomial!

then $\quad a_1 = 0,\ a_2 = 0, \dots\ a_m = 0.$

Example If a set $S \subseteq V$ (V = a vector space) contains $\vec{0}$, then S is linearly dependent:

take $n = 1$, $v_1 = \vec{0}$, $\alpha_1 = 1$. $\quad \vec{0} = 1 \cdot \vec{0} = \alpha_1 \cdot v_1$

Remark 4.0 Suppose V is a vector space, $v_1 \dots v_m \in V$ and there are $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ s.t. $\alpha_m \neq 0$, and $\alpha_1 v_1 + \dots + \alpha_m v_m = \vec{0}$

Then $\operatorname{span}\{v_1 \dots v_m\} = \operatorname{span}\{v_1, \dots v_{m-1}\}$.

Reason: $\operatorname{span}\{v_1, \dots v_{m-1}\} \subseteq \operatorname{span}\{v_1, \dots v_m\}$.

On the other hand, since $\alpha_m \neq 0$, $v_m = \left(-\frac{\alpha_1}{\alpha_m}\right) v_1 + \left(-\frac{\alpha_2}{\alpha_m}\right) v_2 + \dots$

$$+ \left(-\frac{\alpha_{m-1}}{\alpha_m}\right) v_{m-1}.$$

So if $\vec{u} \in \operatorname{span}\{v_1, \dots v_m\}$, $\vec{u} = \beta_1 v_1 + \dots + \beta_m v_m$ for some $\beta_1, \dots \beta_m$

$$= \beta_1 v_1 + \dots + \beta_{m-1} v_{m-1} + \left(-\frac{\alpha_1}{\alpha_m}\right) v_1 + \dots + \left(-\frac{\alpha_{m-1}}{\alpha_m}\right) v_{m-1}$$

$$\in \operatorname{span}\{v_1, \dots v_{m-1}\}.$$

$\therefore \operatorname{span}\{v_1, \dots v_m\} \subseteq \operatorname{span}\{v_1, \dots v_{m-1}\}$.

Definition Let V be a vector space. A subset S of V is a basis if (1) $\operatorname{span} S = V$ and (2) S is linearly independent.

Ex $V = \mathbb{R}^n$ $\quad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}}$ slot $\quad S = \{e_1, \dots e_n\}$ is a basis of $\mathbb{R}^n$.

Ex $V = \mathbb{R}[x]$ $\quad S = \{x^i\}_{i=0}^{\infty}$ is a basis of $\mathbb{R}[x]$.

Lemma 4.1 Let V be a vector space, $\{v_1, \dots v_m\} \subseteq V$ such that $\operatorname{span}\{v_1, \dots v_m\} = V$. Then a subset of $\{v_1, \dots v_m\}$ is a basis of V.

Proof Induction on m = # of elements in $\{v_1, \dots v_m\}$.

Base case: $m = 1$. $V = \operatorname{span}\{v_1\}$. If $v_1 = \vec{0}$, $V = \{\vec{0}\}$. Then $\emptyset \subseteq \{\vec{0}\}$ is a basis of $V = \{\vec{0}\}$.

Suppose $v_1 \neq \vec{0}$. We need to show: $\{v_1\}$ is lin independent.

Suppose $\alpha v_1 = \vec{0}$ for some $\alpha \in \mathbb{R}$. If $\alpha \neq 0$, α^{-1} exists

$\vec{0} = \alpha^{-1} \cdot \vec{0} = \alpha^{-1} \cdot \alpha v_1 = 1 \cdot v_1 = v_1$. Contradiction.

$\therefore \alpha = 0$. $\Rightarrow \{v_1\}$ is lin independent, hence a basis.

Inductive step: Suppose lemma holds for $m = k$, and

$V = \operatorname{span}\{v_1, \dots v_{k+1}\}$.

If $\{v_1, \dots v_{k+1}\}$ is lin. independent, it's a basis.

So suppose $\{v_1, \dots, v_{k+1}\}$ is lin. dependent. Then $\exists\, \alpha_1, \dots \alpha_{k+1}$ not all zero s.t. $\vec{0} = \alpha_1 v_1 + \dots + \alpha_{k+1} v_{k+1}$.

No loss of generality to assume: $\alpha_{k+1} \neq 0$. By Remark 4.0, span$\{v_1, \dots v_{k+1}\}$ = span$\{v_1, \dots v_k\}$.

By inductive assumption, a subset of $\{v_1, \dots v_k\}$ is a basis of V.

$\Rightarrow$ a subset of $\{v_1, \dots, v_k, v_{k+1}\}$ is a basis of V.

We're now done by induction. $\square$

(Important) <u>Lemma 4.2</u> Let V be a vector space. Suppose we have a set of n distinct vectors $\{x_1, \dots x_n\} \in V$ st span$\{x_1, \dots x_n\} = V$ an a set of m distinct vectors $\{y_1, \dots y_m\} \in V$ which is linearly independent. Then $m \le n$.

<u>Proof</u> Note that since $\{y_1, \dots y_m\}$ is linearly independent, $y_i \neq \vec{0}$ $\forall i$.

Since $\{x_1, \dots x_n\}$ spans V, $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{R}$ with

$$y_1 = \alpha_1 x_1 + \dots + \alpha_n x_n.$$

Since $y_1 \neq \vec{0}$, not all α's are zero. We may assume $\alpha_1 \neq 0$.

Then $x_1 = \frac{1}{\alpha_1}(y_1 - \alpha_2 x_2 - \dots - \alpha_n x_n)$.

$\Rightarrow \{y_1, x_2, \dots x_n\}$ spans V (why?)

$\Rightarrow y_2 = \beta_1 y_1 + \beta_2 x_2 + \dots + \beta_n x_n$ for some $\beta_1, \dots \beta_n \in \mathbb{R}$.

If $\beta_2 = 0, \beta_3 = 0, \dots \beta_n = 0$, $y_2 = \beta_1 y_1 \Rightarrow \vec{0} = \beta_1 y_1 + (-1) y_2 + 0 y_3 + \dots + 0 y_m$.

This contradicts linear independence of y_i's since $(-1) \neq 0$.

$\Rightarrow$ One of the $\beta_2, \dots \beta_n$ is nonzero. Say $\beta_2 \neq 0$

Then $$x_2 = \frac{1}{\beta_2}(y_2 - \beta_1 y_1 - \beta_3 x_3 - \dots - \beta_n x_n)$$

$\Rightarrow \{y_1, y_2, x_3, \dots x_n\}$ spans V.

Keep going. The process stops if either

1) we replace all x's with y's and there are no x's or y's left over, i.e. $m = n$.

or

2) we run out of y's before we run out of x's, i.e. $m < n$.

or 3) we run out of x's before we run out of y's i.e. $m > n$ and $\{y_1, \dots y_n\}$ spans V.

$\Rightarrow y_{n+1} \in \text{span}\{y_1, \dots y_n\}$ i.e. $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{R}$ s.t.

$$y_{n+1} = \alpha_1 y_1 + \dots + \alpha_n y_n$$

$$\Rightarrow \vec{0} = \alpha_1 y_1 + \dots + \alpha_n y_n + (-1) y_{n+1} + 0 \cdot y_{n+2} + \dots + 0 \cdot y_m.$$

Which contradicts the independence of $\{y_1, \dots y_m\}$.

So case (3) is impossible. $\Rightarrow$ $m \le n$. □

<u>Theorem 4.3</u> Suppose $\{x_1 \dots x_n\}$, $\{y_1 \dots y_m\}$ are two bases of a vector space V. Then $n = m$.

<u>Proof</u> Since $\{x_1 \dots x_n\}$ spans V and $\{y_1 \dots y_m\}$ is linearly independent $m \le n$ by 4.2.

Since $\{y_1 \dots y_m\}$ spans V and $\{x_1 \dots x_n\}$ is linearly independent $n \le m$.

□