Last time:

- Cayley-Hamilton: $T: V \to V$ linear, $p_T(x) = \det(x\, id_V - T)$
  Then $p_T(T) = 0$
- The minimal polynomial $m_T(x)$ of T is $(x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$ where $\lambda_1 \dots \lambda_k$ are eigenvalues of T and
  $$m_j = \min\{i \mid N((T-\lambda_j\, id)^i) = N((T-\lambda_j\, id)^{i+1})\}$$
- $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$, T-invariant decomposition,
  where $K_{\lambda_j} = N((T-\lambda_j\, id)^{m_j})$

Aside if $V = U \oplus W$, $A: U \to U$, $B: W \to W$ linear
then we have $A \oplus B : U \oplus W \to U \oplus W$
$$(A \oplus B)(u+w) = Au + Bw \quad \forall u \in U, w \in W$$

$\to \quad T = (T|_{K_{\lambda_1}}) \oplus (T|_{K_{\lambda_2}}) \oplus \cdots \oplus (T|_{K_{\lambda_k}})$

Let $N_j := T|_{K_{\lambda_j}} - \lambda_j\, id_{K_{\lambda_j}}$. Then $N_j^{m_j} = 0$

ie N_j is nilpotent, $D_j := \lambda_j\, id_{K_j}$ is diagonalizable.

$\Rightarrow T|_{K_{\lambda_j}} = D_j + N_j$

D_j diagonalizable, N_j nilpotent

and $D_j \circ N_j = N_j \circ D_j$

$\Rightarrow$ Now let $D = D_1 \oplus \cdots \oplus D_k$
$N = N_1 \oplus \cdots \oplus N_k$

Then (1) $T = D + N$

(2) D is diagonalizable, N nilpotent and

(3) $D \circ N = N \circ D$.

This is not quite Jordan normal theorem, but it's closed.
For some applications, it's enough.

What's left to prove is

Theorem 37.1 (see Treil, Theorems 4.1, 4.2 in the last chapter)

$A : W \to W$ nilpotent linear map. Then there is a basis $\mathcal{B}$ of W so that $[A]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} J_1(0) & & 0 \\ & \ddots & \\ 0 & & J_r(0) \end{pmatrix}$ where $J_i(0) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$J_i(0)$ are Jordan blocks with zeros on the diagonal.

Proof sadly we're out of time...

Note what Thm 37.1 says in the case where there is only one block:

$\exists$ a basis $b_1, b_2, \dots b_k$ of W s.t.

$A b_k = b_{k-1}$, $A b_{k-1} = b_{k-2}, \dots$ $A b_2 = b_1$ and $A b_1 = 0$.

Procedure for computing J.N.F. of $T : V \to V$

① compute $p_T(x) = \det(x \, \mathrm{Id}_V - T)$ and factor it:

$$p_T(x) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$$

② For each i, $1 \le i \le k$, compute $N((T-\lambda_i id))$, $N((T-\lambda_i id)^2)$ ...

... until $N((T-\lambda_i id)^{m_i}) = N((T-\lambda_i id)^{m_i+1})$

Set $K_{\lambda_i} = N((T-\lambda_i id)^{m_i})$, the λ_i generalized eigenspace

m_i = size of the largest λ_i - Jordan block, $n_i = \dim K_{\lambda_i}$

$E_{\lambda_i} := N(T - \lambda_i id)$ the λ_i-eigenspace of T

$\dim E_{\lambda_i}$ = geometric multiplicity of λ_i = # of λ_i Jordan blocks

while $n_i = \dim K_{\lambda_i}$ = algebraic multiplicity of λ_i

= sum of dimensions of λ_i Jordan blocks

Note $d_i^{(1)} = \dim N(T - \lambda_i id) = \dim E_{\lambda_i}$ = # of Jordan blocks

$d_i^{(2)} = \dim N((T-\lambda_i id)^2)$ = # of Jordan blocks of size ≥ 2

$\vdots$ $\qquad$ $\vdots$ $\qquad$ $\vdots$

Example $T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$ $\qquad$ $\det(xI - T)$

$\det(xI - T) = \det\begin{pmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ -1 & 0 & x-2 & 0 \\ 0 & -1 & 0 & x-2 \end{pmatrix} = (x-2)^4 \Rightarrow K_2(T) = \mathbb{C}^4$

$$N(T-2I) = N\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \text{span}\{e_3, e_4\}$$

$\Rightarrow$ two Jordan blocks

$$(T-2I)^2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ 1 & 0 & \\ 0 & 1 & 0 \end{array}\right)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & 0 \\ 1 & 0 & \\ 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\Rightarrow$ There is no chain of vectors $b_3 \xrightarrow{T} b_2 \xrightarrow{T} b_1 \to 0$ with $b_3, b_2, b_1 \neq 0$

$\Rightarrow$ each Jordan block has size ≤ 2. We have 2 blocks and their dimensions add up to 4. $\Rightarrow$

$$JNF(T) = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right). \Leftrightarrow JNF(T-2I) = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $b_1 = e_3$, $b_3 = e_4$. Need to find b_2, b_4 st

$(T-2I)(b_2) = b_1$, $(T-2I)b_4 = b_3$.

i.e. solve

$$\underbrace{\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{b_2} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{b_1} \qquad \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & \\ \hline 1 & 0 & 0 \\ 0 & 1 & \end{array}\right)\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{b_4} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{b_3}$$

By inspection: $b_2 = e_1$ $\qquad b_4 = e_2$

So $\mathcal{B} = \{e_3, e_1, e_4, e_2\}$. $\rightsquigarrow B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Now compute B^{-1} by considering $\left(B \,\middle|\, \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right)$ and using row operations on B to convert it to I. Or $B^{-1} = B^T$ since B is orthogonal.

We get $B^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Now check that

$$B^{-1}TB = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

$\square$

Why care about J.N.F?

Suppose we want to solve a linear system of ODE's

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = A\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad x_1(0) = x_1^{(0)},\ x_2(0) = x_2^{0} \ldots$$

$$x(t) = e^{tA}\begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$$ is the unique solution.

We need to compute $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$.

<u>Fact</u> (ie a theorem stated without proof) Suppose $X, Y \in M_{n,n}(\mathbb{C})$ and $X \circ Y = Y \circ X$. Then

$$e^{X+Y} = e^X \circ e^Y.$$

Now, Jordan normal form $\Rightarrow$ $A = D + N$, D diagonalizable, N nilpotent. $D \circ N = N \circ D$. Now fact $\Rightarrow$

$$e^{tA} = e^{tD} e^{tN}$$

Since N is nilpotent $\exists m$ st $N^m = 0$.

$$\Rightarrow e^{tN} = I + tN + \frac{t^2}{2!}N^2 + \dots + \frac{t^{m-1}}{(m-1)!}N^{m-1} + \underline{0}$$

ie. e^{tN} is a <u>polynomial</u> in t.

D diagonalizable $\Rightarrow \exists$ a matrix B st $B^{-1}DB = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}$

$$\Rightarrow D = B\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix}B^{-1}$$

$$\Rightarrow D^k = B\begin{pmatrix} d_1^k & & \\ & \ddots & \\ & & d_r^k \end{pmatrix}B^{-1}. \Rightarrow e^{tD} = B\begin{pmatrix} e^{td_1} & & 0 \\ & \ddots & \\ 0 & & e^{td_k} \end{pmatrix}B^{-1}.$$

$$\Rightarrow e^{tA} = B\begin{pmatrix} e^{td_1} & & \\ & \ddots & \\ & & e^{td_r} \end{pmatrix}B^{-1} \cdot \begin{pmatrix} \text{matrix w. polynomial} \\ \text{entries in } t \end{pmatrix}$$

We may be able to say more.

For example: Suppose $\mathrm{Re}(d_j) < 0 \ \forall j$

<u>Fact</u> Suppose $\mathrm{Re}(d) < 0$. Then $\forall$ polynomial $p(t)$

$$e^{td} p(t) \xrightarrow[t \to \infty]{} 0.$$

$\Rightarrow$ If all eigenvalues of A have negative real parts then then for any solution $x(t)$ of $\frac{d}{dt}x = Ax$

$$x(t) \xrightarrow[t \to \infty]{} 0$$