Last time $T: V \to V$ linear, V finite dim complex vector space.

$$\det(x\, \mathrm{id}_V - T) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k}$$

Then

(i) $\forall j \ \exists m_j$ so that $N((T-\lambda_j \mathrm{id})^{m_j - 1}) \subsetneq N((T-\lambda_j \mathrm{id})^{m_j})$

and $N((T-\lambda_j \mathrm{id})^{m_j}) = N((T-\lambda_j \mathrm{id})^{m_j+r})$ for all $r \geq 1$.

We set

$$K_{\lambda_j} = K_{\lambda_j}(T) = N((T-\lambda_j \mathrm{id})^{m_j})$$

the λ_j-generalized eigenspace of T.

(ii) $\Rightarrow$ $V = K_{\lambda_1}(T) \oplus W$

direct sum decomposition into <u>T-invariant</u> subspaces

(iii) The characteristic polynomial of $T|_{K_{\lambda_1}}$ is $(x-\lambda_1)^{n_1}$

(so $\dim K_{\lambda_1} = n_1 \equiv$ algebraic multiplicity of λ_1)

(iv) The characteristic polynomial of $T|_W$ is

$$(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$$

(v) To prove (iii) we proved first: λ_1 is the only eigenvalue of $T|_{K_{\lambda_1}}$.

(vi) <u>Note also:</u> (i) $\Rightarrow$ $(T-\lambda_1 \mathrm{id})|_{K_{\lambda_1}}$ is <u>nilpotent</u> with

$$((T-\lambda_1 \mathrm{id})|_{K_{\lambda_1}})^{m_1} = 0, \quad ((T-\lambda_1 \mathrm{id})|_{K_{\lambda_1}})^{m_1 - 1} \neq 0.$$

Since $(T-\lambda_1 \mathrm{id})|_{K_{\lambda_1}} = T|_{K_{\lambda_1}} - \lambda_1 \mathrm{id}_{K_{\lambda_1}}$

the minimal polynomial of $T|_{K_{\lambda_1}}$ is $(x-\lambda_1)^{m_1}$.

<u>Note</u> $m_1 \leq \dim K_{\lambda_1} = n_1$

(vii) Same argument shows: - $\dim K_{\lambda_j} = n_j \ \forall j$

- $\exists m_j$ s.t. $(T|_{K_{\lambda_j}} - \mathrm{id}_{K_{\lambda_j}})^{m_j} = 0 \quad (T|_{K_{\lambda_j}} - \mathrm{id}_{K_{\lambda_j}})^{m_j - 1} \neq 0$

and $m_j \leq \dim K_{\lambda_j} = n_j$.

Lemma 36.1 Let $T: V \to V$, $\lambda_1, \ldots \lambda_k$, $K_{\lambda_1}, \ldots K_{\lambda_k}$ be as above.

$$V = K_{\lambda_1} \oplus W.$$

Then
$$W = K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k},$$
a decomposition into T-invariant subspaces.

Proof By induction

$$W = K_{\lambda_2}(T|_W) \oplus \cdots \oplus K_{\lambda_k}(T|_W).$$

We now argue that $K_{\lambda_j}(T|_W) = K_{\lambda_j}(T) \quad \forall j \geq 2$.

Note $\dim K_{\lambda_j}(T|_W) = n_j$

(since the char poly of $T|_W$ is $(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$)

Also, if $v \in K_{\lambda_j}(T|_W)$, $\exists \ell$ st

$$0 = (T|_W - \lambda_j \, id_W)^{\ell}(v) = (T - \lambda_j \, id_V)^{\ell}(v) \qquad (\text{since } v \in W \subseteq V)$$

$$\Rightarrow \quad K_{\lambda_j}(T|_W) \subseteq K_{\lambda_j}(T).$$

Since $\dim K_{\lambda}(T|_W) = n_j = \dim K_{\lambda_j}(T)$

$$K_{\lambda_j}(T|_W) = K_{\lambda_j}(T). \qquad \square$$

Corollary 36.2 Let $T: V \to V$, $\lambda_1, \ldots \lambda_k$ etc be as above.

Then (1) $(T - \lambda_1 id)^{n_1} \circ \cdots \circ (T - \lambda_k id)^{n_k} = 0$ (Cayley-Hamilton)

(2) The minimal polynomial $m_T(x)$ of T is

$$m_T(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}$$

(where, as above $(T - \lambda_j id)^{m_j}|_{K_{\lambda_j}} = 0$, $(T - \lambda_j id)^{m_j - 1}|_{K_{\lambda_j}} \neq 0$)

Proof Suppose $v \in K_{\lambda_j}$ for some j. Then,

$$(T - \lambda_j id)^{m_j}(v) = 0$$

Since $m_j \leq n_j$, $(T - \lambda_j id)^{n_j}(v) = 0$.

$$\Rightarrow P_T(T)v = (T - \lambda_1 id)^{n_1} \circ \cdots \circ (T - \lambda_k id)^{n_k}(v)$$
$$= \left(\prod_{i \neq j} (T - \lambda_i id)^{n_i}\right) \circ (T - \lambda_j id)^{n_j}(v)$$

$$= \left(\prod_{i \neq j} (T - \lambda_i id)^{n_i}\right)(0) = 0$$

Since $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_n}$, $P_T(T)v = 0 \ \forall v \in V$.

Since $p_T(T) = 0$, $p_T(x) \in N(ev_T) = m_T(x) \cdot \mathbb{C}[x]$

$\Rightarrow$ $p_T(x) = h(x)\, m_T(x)$ for some $h \in \mathbb{C}[x]$

$p_T(x) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k} \Rightarrow m_T(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

for some $r_1, \dots r_n$, $0 \le r_i \le n_i$.

By def of m_j, $\left(T - \lambda_j\, id\right)^{m_j - 1}\big|_{K_{\lambda_j}} \neq 0$, $\left(T - \lambda_j\, id\right)^{m_j}\big|_{K_{\lambda_j}} = 0$

Moreover $\left(T - \lambda_i\, id\right)\big|_{K_{\lambda_j}}$ is <u>invertible</u> $\forall\, i \neq j$.

It follows that

$$m_T(x) = (x-\lambda_1)^{m_1} \cdots (x-\lambda_k)^{m_k}.$$

Given a T-invariant decomposition

$$V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$$

a choice of basis $\mathcal{B}_i$ for K_{λ_i} gives us a basis

$$\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k \quad \text{so that}$$

$$[T]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \boxed{T_1} & & 0 \\ 0 & \boxed{T_2} & \\ 0 & & \ddots \\ & & & \boxed{T_k} \end{pmatrix}$$

where $T_i = \left[T|_{K_{\lambda_i}} \right]_{\mathcal{B}_i \mathcal{B}_i}$.

To prove existence of JNF remains to show:

Suppose $T: V \to V$ is linear, λ is the only eigenvalue of T. (so $V = K_\lambda$ and $T - id$ is nilpotent).

Then $\exists$ a basis $\mathcal{B}$ of V st

$$(*) \quad [T]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} J_1(\lambda) & & \\ & \ddots & \\ & & J_s(\lambda) \end{pmatrix}$$

where $J_i(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \\ & & & \lambda \end{pmatrix}$, Jordan blocks w eigenvalue λ.

<u>Note</u> $(*)$ holds $\Leftrightarrow$

$$[T - \lambda\, id]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} J_1(0) & & 0 \\ 0 & \ddots & \\ & & J_s(0) \end{pmatrix}$$

$J_i(0) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 1 \\ & & & 0 \end{pmatrix}$

We'll prove existence of such a basis $\mathcal{B}$ on Monday.

See Thm 4.1, Thm 4.2 pp 266 – 272 of Treil.

<u>Example</u> $T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 1/2 & 0 & 2 \end{pmatrix}$ J.N.F? basis $\mathcal{B}$ st $[T]_{\mathcal{B}\mathcal{B}} =$ JNF?

<u>Solution</u>

$$\det(xI - T) = \det\begin{pmatrix} x-2 & 0 & 0 \\ 0 & x-2 & -1 \\ -1/2 & 0 & x-2 \end{pmatrix} = (x-2)\det\begin{pmatrix} x-2 & -1 \\ 0 & x-2 \end{pmatrix} + 0 + 0$$

(expansion by top row)

$= (x-2)^3$. $\Rightarrow$ 2 is the only eigenvalue $\Rightarrow$

JNF is $\begin{pmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 2 \end{pmatrix}$ or $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$ or $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$T - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}$. $\Rightarrow \dim R(T-2I) = 2$ and $\dim N(T-2I) = 1$

$\Rightarrow e_2$ spans $N(T-2I)$. $\Rightarrow$ JNF $= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

<u>basis?</u> We need $\mathcal{B} = \{b_1, b_2, b_3\}$ s.t.

$[(T-2I)]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ie. $(T-2I)b_1 = 0$

$(T-2I)b_2 = b_1$

$(T-2I)b_3 = b_2$.

We take $b_1 = e_2$, which spans $N(T-2I)$. $b_2 \in R(T-2I)$

since $b_2 = (T-2I)b_3$. $R(T-2I) = \operatorname{span}\{e_3, e_2\}$.

$(T-2I)e_3 = e_2 = b_1$ so let $b_2 = e_3$.

Now we need b_3 st $(T-2I)b_3 = e_3$, ie $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix} b_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let $b_3 = 2e_1$.

<u>Note</u> $[I]_{SB} = (b_1 | b_2 | b_3) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. and $[I]_{SB}^{-1}\, T\, [I]_{SB} = [T]_{BB}$.

One can check that $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}$ (start with $\left(\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(I \,\middle|\, \cdots \right)$)

and

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 1/2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$