Last time:

Stated without proof (1) Cayley - Hamilton Theorem:

Given $T: V \to V$ ($\dim V < \infty$) $\quad p_T(T) = 0$ where

$p_T(x) = \det(x \, id_V - T)$, the <u>characteristic</u> polynomial of T

(2) There exists a unique polynomial $m_T(x)$,

the <u>minimal</u> polynomial of T so that

$m_T(x) = 1 \cdot x^n + a_{n-1} x^{n-1} + \cdots + x_0$ ie m_T is <u>monic</u> and

$h(T) = 0 \Rightarrow \exists q(x) \in \mathbb{C}[x]$ s.t. $h(x) = m_T(x) q(x)$

ie m_T divides any $h(x) \in N(ev_T)$.

We've seen: characteristic polynomial need not be minimal.

Lemma 35.1 Suppose $\dim V < \infty$, $S: V \to V$ is linear. Then

1) $\exists m \geq 1$ so that $N(S^m) = N(S^{m+1})$

2) if $N(S^m) = N(S^{m+1})$ then $N(S^m) = N(S^{m+r}) \ \forall r \geq 1$

3) $V = N(S^m) \oplus R(S^m)$ and (4) $N(S^m)$, $R(S^m)$ are S-invariant.

Proof (1) $N(S) \subseteq N(S^2) \subseteq N(S^3) \subseteq \cdots \subseteq N(S^k) \subseteq \cdots$

$\dim N(S) \leq \dim N(S^2) \leq \dim N(S^3) \leq \cdots \leq \dim N(S^k) \leq \dim V$

$\Rightarrow$ the set of integers $\{\dim N(S^i)\}_{i=1}^{\infty}$ is <u>finite</u>

$\Rightarrow \exists m$ st $\dim N(S^m) = \dim N(S^{m+1})$. $\to N(S^m) = N(S^{m+1})$

(2) If $v \in N(S^{m+2})$ then $0 = S^{m+1}(S(v))$. $\Rightarrow S(v) \in N(S^{m+1}) = N(S^m)$

$\Rightarrow 0 = S^m S(v)$. $\Rightarrow v \in N(S^{m+1})$

$\Rightarrow N(S^{m+2}) \subseteq N(S^{m+1}) = N(S^m)$

Similarly $N(S^{m+3}) \subseteq N(S^{m+2}) \ldots$

$\therefore$ For all $r \geq 0$ $\quad N(S^{m+r}) = N(S^m)$.

(3) If $v \in N(S^m) \cap R(S^m)$ then $v = S^m(w)$ for some w and

$0 = S^m(v)$. $\Rightarrow 0 = S^m S^m(w)$. $\to w \in N(S^{m+m}) = N(S^m)$

$\Rightarrow 0 = S^m(w) = v$. $\quad \therefore N(S^m) \cap R(S^m) = 0$

Since $\dim V = \dim N(S^m) + \dim R(S^m)$, $\quad V = N(S^m) \oplus R(S^m)$.

Finally $\forall v \in N(S^m)$ $\quad S^m(Sv) = S(S^m v) = S(0) = 0 \Rightarrow S(v) \in N(S^m)$

$\Rightarrow S(N(S^m)) \subseteq N(S^m)$.

If $u \in R(S^m)$ $\quad u = S^m w$ for some w. $\Rightarrow Su = SS^m w = S^m(\underline{Sw})$

$\in R(S^m)$. $\Rightarrow \quad S(R(S^m)) \subseteq R(S^m)$. $\square$

Now consider $T: V \to V$. and suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T. Let $S = T - \lambda\, id_V$.

Then $N(S) \neq 0$. Also $\exists m$ st $N(S^m) = N(S^{m+1}) = N(S^{m+2}) \ldots$

We set $K_\lambda = N(S^m) = N((T - \lambda\, id_V)^m)$.

Then $\forall n \geq m$ $\quad N((T-\lambda\, id_V)^n) = K_\lambda$.

$\rightarrow K_\lambda = \bigcup_{\forall k=1}^{\infty} N((T - \lambda\, id_V)^k)$,

<u>The generalized λ-eigenspace of T.</u>

<u>Remark</u> $\forall$ subspace $U \subseteq V$, $\quad T(U) \subseteq U \Leftrightarrow S(U) \subseteq U$ where $S = T - \lambda\, id_V$.

<u>Reason</u> $Tu \in U \ \forall u \in U \Longleftrightarrow S(u) = Tu - \lambda u \in U$.

Lemma 35.1 $\Rightarrow$

<u>Lemma 35.2</u> Let $T: V \to V$ be linear, λ = eigenvalue of T.

Then $\quad V = K_\lambda \oplus W$

where W is a T-invariant and $T - \lambda\, id_V|_W$ is <u>injective</u>

<u>Proof</u> $K_\lambda = N(S^m)$, $W = R(S^m)$ where

$S = T - \lambda\, id_V$ and m is the <u>smallest</u> integer st.

$N(S^m) = N(S^{m+1})$.

If $\quad v \in N(T - \lambda\, id_V) = N(S)$, then $v \in N(S^m)$

But $N(S^m) \cap \underbrace{R(S^m)}_{=W} = 0$

$\Rightarrow N((T - \lambda\, id)|_W) = \{0\}$. $\Rightarrow T - \lambda\, id_V$ is injective

Lemma 35.3 Let $T: V \to V$ be linear, λ eigenvalue of T, $V = K_\lambda \oplus W$ the T-invariant decomposition of Lemma 35.2. Then

1) λ is the only eigenvalue of $T|_{K_\lambda}$.
2) λ is not an eigenvalue of $T|_W$.

Proof (1) Suppose of $v \in K_\lambda$ and $Tv = \mu v$ for some $\mu \in \mathbb{C}$.
Let $S = T - \lambda\, id_V$. Then $Sv = Tv - \lambda v = (\mu - \lambda)v$.
By def of K_λ, $\exists m$ s.t. $0 = S^m v$. On the other hand
$S^m v = (\mu - \lambda)^m v$. $\Rightarrow 0 = (\mu - \lambda)^m v$ and $v \neq 0$.
$\Rightarrow \mu = \lambda$.

(2) λ is an eigenvalue of $T|_W \Leftrightarrow \exists\, 0 \neq w \in W$ s.t.
$Tw = \lambda w \Leftrightarrow w \in N((T - \lambda\, id_V))$
But $N((T - \lambda\, id_V)|_W) = \{0\}$.
So λ cannot be an eigenvalue of $T|_W$. ☐

Lemma 35.4 Suppose
$$C = \begin{array}{cc} \begin{array}{cc} \overbrace{}^{k} & \overbrace{}^{l} \end{array} & \\ \left(\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right) & \begin{array}{c} k \\ l \end{array} \end{array}$$
is a $(k+l)\times(k+l)$ block diagonal matrix. Then
$$\det C = \det A \cdot \det B.$$

Proof Note first that
$$\left(\begin{array}{c|c} A & O \\ \hline O & B \end{array}\right) = \left(\begin{array}{c|c} A & O \\ \hline O & I_l \end{array}\right) \cdot \left(\begin{array}{c|c} I_k & O \\ \hline O & B \end{array}\right)$$

Next observe that
$$D: \mathbb{R}^{k^2} \to \mathbb{R},\ D(A) = \det\left(\begin{array}{c|c} A & O \\ \hline O & I_l \end{array}\right)$$
is k-linear, alternating and
$$D(I_k) = \det\left(\begin{array}{c|c} I_k & O \\ \hline O & I_l \end{array}\right) = 1.$$
$$\Rightarrow \det\left(\begin{array}{c|c} A & O \\ \hline O & I_l \end{array}\right) = \det A.$$
Similarly $\det\begin{pmatrix} I_k & O \\ O & B \end{pmatrix} = \det B$

$$\Rightarrow \det C = \det\left(\begin{array}{c|c} A & O \\ \hline O & I_l \end{array}\right) \cdot \det\left(\begin{array}{c|c} I_k & O \\ \hline O & B \end{array}\right) = \det A \det B.$$

Lemma 35.6 Suppose $T: V \to V$ is a linear map, $p_T(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$ the characteristic polynomial of T and $V = K_{\lambda_1} \oplus W$ the T-invariant decomposition of V as above. Then

$$p_{T|K_\lambda}(x) = (x-\lambda_1)^{n_1} \qquad p_{T|W} = (x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$$

and, in particular, $\dim K_{\lambda_1} = n_1$.

Proof Pick a basis $\mathcal{A}$ of K_{λ_1}, $\mathcal{B}$ of W. Then $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ is a basis of V and

$$[T]_{\mathcal{C}\mathcal{C}} = \left(\begin{array}{c|c} [T|_{K_{\lambda_1}}]_{\mathcal{A}\mathcal{A}} & 0 \\ \hline 0 & [T|W]_{\mathcal{B}\mathcal{B}} \end{array}\right)$$

$$p_T(x) = \det(x I_n - [T]_{\mathcal{C}\mathcal{C}}) = \det(x I_r - [T|_{K_{\lambda_1}}]_{\mathcal{A}\mathcal{A}}) \cdot \det(x I_l - [T|W]_{\mathcal{B}\mathcal{B}})$$

where $n = \dim V$, $r = \dim K_\lambda$, $l = \dim W$.

$$\Rightarrow p_T(x) = p_{T|K_\lambda}(x) \cdot p_{T|W}(x).$$

- The only eigenvalue of $T|_{K_{\lambda_1}}$ is λ_1. $\Rightarrow p_{T|K_\lambda}(x) = (x-\lambda_1)^r$
- λ_1 is <u>not</u> an eigenvalue of $T|_W$.

$$(x-\lambda_1)^{n_1} \cdots (x-\lambda_k)^{n_k} = (x-\lambda_1)^r \cdot p_{T|W}(x)$$

$$\Rightarrow r = n_1 \ \& \ p_{T|W} = (x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}.$$

Corollary 35.7 Let $T: V \to V$ be as above. Then

$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$ T-inv decomposition

$\dim K_{\lambda_i} = n_i$, the algebraic multiplicity of λ_i.

$T|_{K_{\lambda_i}}$ has only λ_i as its eigenvalue and

$$(T|_{K_{\lambda_i}} - \lambda_i \cdot \mathrm{Id}_{K_{\lambda_i}})^{m_i} = 0 \quad \text{for some } m_i \le n_i.$$