04/22/2019
Mo

Last time: • Finished proving the spectral theorem for Hermitian operators

• defined Jordan blocks, Jordan normal form of a linear map $T: V \to V$.

Checked that $ev_T : \mathbb{C}[x] \to Hom(V,V)$

$$ev_T\left(\sum_{j=0}^{n} a_j x^j\right) = \sum_{j=0}^{n} a_j T^j$$

is $\mathbb{C}$-linear and multiplicative $ev_T(pq) = ev_T(p) \circ ev_T(q)$ for all polynomial $p, q \in \mathbb{C}[x]$.

Note: Since $pq = qp$ $ev_T(p) \circ ev_T(q) = ev_T(pq) = ev_T(qp) =$
$= ev_T(q) \circ ev_T(p)$

Context An algebra over $\mathbb{C}$ is a vector space A together with a bilinear map ("multiplication")

$$A \times A \to A, \quad (a_1, a_2) \mapsto a_1 a_2.$$

Ex $\mathbb{C}[x]$, $Hom(V,V)$, $M_{n,n}(\mathbb{C})$ are algebras $/\mathbb{C}$.

Note: 1) We don't require the "multiplication" to be associative

2) If multiplication is associative, A is an associative algebra.

3) If A is an associative algebra and we forget scalar multiplication, we get a ring.

Ex $\mathbb{C}[x]$, $Hom(V,V)$, $M_{n,n}(\mathbb{C})$ are associative algebras and rings.

Ex $[\cdot,\cdot] : M_{n,n}(\mathbb{C}) \times M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$

$$[A, B] := AB - BA$$

is bilinear but not associative

Lemma 34.1 Suppose A is an algebra. Then $\forall a \in A$

$$a \cdot 0 = 0 \cdot a = 0.$$

Proof $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$. Now add $-(a \cdot 0)$ to both sides.

Similarly $0 \cdot a = 0$. $\square$

Definition Let A and B be two algebras over $\mathbb{C}$. A linear map $f: A \to B$ is an algebra homomorphism if f preserves multiplication : $f(a_1 a_2) = f(a_1) f(a_2)$ $\forall a_1, a_2 \in A$.

Ex $ev_T : \mathbb{C}[x] \to \mathrm{Hom}(V,V)$ is an algebra homomorphism.

Remark One can show : if $f: A \to B$ is a bijective algebra homomorphism then $f^{-1}: B \to A$ is also an algebra homomorphism. Invertible algebra homomorphisms are called isomorphisms.

Ex Let V be a finite dim vector space. Choose a basis B of V. Then the map

$$\mathrm{Hom}(V,V) \to M_{n,n}(\mathbb{C}) \quad (n = \dim V)$$

$$T \mapsto [T]_{BB}$$

is an algebra isomorphism. (check that!)

Lemma 34.2 Let $f: A \to B$ be an algebra homomorphism. Then $\forall a \in A$, $\forall x \in N(f) = \{x \in A \mid f(x) = 0\}$. $ax, xa \in N(f)$.

Proof $f(ax) = f(a) f(x) = f(a) \cdot 0 = 0$ by 34.1
$\Rightarrow ax \in N(f)$. Similarly $xa \in N(f)$

Def A subspace I of an algebra A is an ideal if $\forall x \in I$ $\forall a \in A$ $ax, xa \in I$.

Theorem 34.3 Let $I \subseteq \mathbb{C}[x]$ be an ideal. There exists a unique monic polynomial $m \in \mathbb{C}[x]$ so that

$$I = \{ q \cdot m \mid q \in \mathbb{C}[x] \} = m \cdot \mathbb{C}[x]$$

Proof omitted. We'll do it in math 427.
One takes m to be the polynomial of the smallest degree in I.

<u>Consequence</u> $\forall\ T: V \to V$ there exists a unique monic polynomial $m_T \in \mathbb{C}[x]$ st

$$N(ev_T) = m_T \cdot \mathbb{C}[x].$$

ie. $\forall p \in \mathbb{C}[x]$, $p(T) = 0 \Rightarrow p(x) = m_T(x) \cdot q(x)$ for some $q(x) \in \mathbb{C}[x]$

m_T is called the <u>minimal</u> polynomial of T

<u>Theorem</u> (Cayley-Hamilton) Let $T: V \to V$ be a linear map. The characteristic polynomial p_T of T lies in $N(ev_T)$ (ie. $p_T(T) = 0$)

Hence by 34.3, $p_T(x) = q(x)\, m_T(x)$ for some $q(x) \in \mathbb{C}[x]$

<proof postponed>.

<u>Ex</u> $T = \begin{pmatrix} 8 & 1 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ $\quad p_T(x) = \det(xI - T)$

$$= \det\begin{pmatrix} x-8 & -1 & 0 \\ 0 & x-8 & 0 \\ 0 & 0 & x-8 \end{pmatrix} = (x-8)^3.$$

Note that $8 \cdot I - T = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \neq 0$

and $(8I - T)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So $m_T(x) = (x-8)^2$

In general $\forall T$, $p_T(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$

where $\lambda_1, \ldots \lambda_k$ are eigenvalues of T

$n_1, \ldots n_k$ are called the <u>algebraic</u> multiplicities of the eigenvalues.

$\dim\{v \in V \mid Tv = \lambda_k v\}$ is the <u>geometric</u> multiplicity of λ_k.

In ex If $T = \left(\begin{array}{cc|c} 8 & 1 & 0 \\ 0 & 8 & 0 \\ \hline 0 & 0 & 8 \end{array}\right)$ algebraic mult of 8 is 3

geometric mult of 8 is <u>2</u>

Recall 1) For a linear map $T : V \to V$ a subspace $U \leq V$ is **T-invariant** if $\forall u \in U$, $T(u) \in U$ (**not!** $T(u) = u$)

2) if $V = U \oplus W$, U, W are T-invariant $\{a_1 \ldots a_n\}$ basis of U, $\{b_1 \ldots b_k\}$ basis of W, and $\mathcal{E} = \{a_1 \ldots a_n, b_1 \ldots b_k\}$

Then
$$[T]_{\mathcal{E}\mathcal{E}} = \left(\begin{array}{c|c} \ast & 0 \\ \hline \underbrace{0}_{n} & \underbrace{\ast}_{k} \end{array}\right) \begin{matrix} \} n \\ \} k \end{matrix}$$

So blocks in Jordan normal form of T must come from a decomposition into T-invariant subspaces

Recall if $\dim V < \infty$ and $S : V \to V$ is linear, Then $\exists k$ s.t.
$$N(S^k) = N(S^{k+1})$$
and
$$N(S) \subsetneq N(S^2) \subsetneq \ldots \subsetneq N(S^k).$$
Moreover, for any $l \geq 1$, $N(S^k) = N(S^{k+l})$.

This is because $v \in N(S^{k+2}) \Rightarrow S^{k+1}(Sv) = 0 \Rightarrow Sv \in N(S^{k+1}) = N(S^k)$
$\Rightarrow 0 = S^k(Sv) = S^{k+1}(v). \Rightarrow v \in N(S^{k+1})$.

Similarly $N(S^{k+2}) = N(S^{k+3}) = N(S^{k+4}) \ldots$
$$\Rightarrow \quad N(S^k) = \bigcup_{j=1}^{\infty} N(S^j)$$

Def Let $T : V \to V$ be a linear map, λ an eigenvalue of T.
The λ-generalized eigenspace of T is
$$K_\lambda := \bigcup_{j=1}^{\infty} N\left((\lambda \mathrm{Id}_V - T)^j\right)$$

Note 1) $K_\lambda = N\left((\lambda \mathrm{Id}_V - T)^k\right)$ for some $k \geq 1$, hence a subspa

2) $\forall v \in N\left((\lambda \mathrm{id}_V - T)^k\right)$
$$(\lambda \mathrm{Id}_V - T)^k (Tv) \underset{\uparrow}{=} T\left((\lambda \mathrm{id}_V - T)^k v\right) = T(0) = 0$$
polynomials in T commute in $\mathrm{Hom}(V, V)$

$\therefore K_\lambda$ is a T-invariant subspace of V.

Recall if $N(S^k) = N(S^{k+1})$ then $V = N(S^k) \oplus R(S^k)$
$\Rightarrow K_\lambda$ has a T-invariant complement $R\left((\lambda \mathrm{id}_V - T)^k\right)$.