Last time

Theorem (Schur) For any linear map $T: V \to V$ on an inner product space V $\exists$ an orthonormal basis $\mathcal{B}$ so that the matrix $[T]_{\mathcal{B}\mathcal{B}}$ is upper triangular.

Cor $\forall A \in M_{n,n}(\mathbb{C})$ $\exists$ unitary matrix U s.t.

$A - UAU^{-1}$ is upper triangular.

Started proving: Spectral theorem:

$T: V \to V$ self adjoint (i.e. $T^* = T$). Then

(1) all eigenvalues of T are real

(2) $\exists$ an orthonormal basis $\mathcal{B}$ of V consisting of eigenvectors of T.

Proved 32.2: $T: V \to V$ linear, $\mathcal{B}$ orthonormal basis of V. Then $[T^*]_{\mathcal{B}\mathcal{B}} = ([T]_{\mathcal{B}\mathcal{B}})^*$.

Proof of (2) of spectral theorem: By Schur's theorem $\exists$ an orthonormal basis $\mathcal{B}$ of V s.t. $A := [T]_{\mathcal{B}\mathcal{B}}$ is upper triangular matrix. Then

$$A^* = ([T]_{\mathcal{B}\mathcal{B}})^* \underset{\uparrow 32.2}{=} [T^*]_{\mathcal{B}\mathcal{B}} \underset{\uparrow \text{since } T = T^*}{=} [T]_{\mathcal{B}\mathcal{B}} = A.$$

Since A is upper triangular and $A = A^*$, A^* is upper triangular.

Since $A^* = (\bar{A})^T$, A^* is also lower triangular.

A matrix that's both upper and lower triangular is diagonal.

Corollary 33.1 Suppose $A \in M_{n,n}(\mathbb{C})$ and $A = A^*$. Then $\exists$ a unitary matrix U st $U^{-1}AU$ is diagonal.

Proof $\exists$ $\mathcal{B} = \{b_1 \ldots b_n\}$ s.t. $U = (b_1 | \ldots | b_n)$ is unitary and $[A]_{\mathcal{B}\mathcal{B}}$ is diagonal.

But $[A]_{\mathcal{B}\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}} A [I]_{\mathcal{A}\mathcal{B}} = U^{-1}AU$.

Jordan normal form

Definition A Jordan block is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 \cdots & 0 \\ 0 & 0 & \lambda & 1 \; 0 \cdots & 0 \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ & \bigcirc & & & \lambda \end{pmatrix}$$

where $\lambda \in \mathbb{C}$.

Ex (λ), $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

are Jordan blocks of size 1, 2, 3, 4.

Definition A matrix A is in Jordan normal form if looks like

$$\begin{pmatrix} J_1 & 0 & 0 & & \\ 0 & J_2 & 0 & & \\ 0 & 0 & J_3 & & \\ & & & \ddots & \\ & & & & J_k \end{pmatrix}$$

where $J_1, \ldots J_k$ are Jordan blocks of possibly different sizes corresponding to possibly different λ's.

Ex

$$A = \left(\begin{array}{cc|c} 2i+1 & 1 & 0 \\ 0 & 2i+1 & 0 \\ \hline 0 & 0 & 3 \end{array}\right)$$ is in Jordan normal form

$$B = \begin{pmatrix} 2i & 1 & 0 \\ 0 & 2i+1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ is not in Jordan normal form.
$(2i \neq 2i+1)$

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$ is in Jordan normal form.

$\left(\begin{array}{c|cc} \boxed{3} & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 0 & 3 \end{array}\right)$ are the two blocks.

Theorem (Jordan normal form / Jordan canonical form)
Let $T: V \to V$ be a linear map on a finite dim v. space $V/\mathbb{C}$. There exists a basis $\mathcal{B}$ of V so that $[T]_{\mathcal{B}\mathcal{B}}$ is in Jordan normal form.
(It will take us a week or so to prove it)

Definition Let $T: V \to V$ be a linear map. A subspace $W \le V$ is *T-invariant* if $T(W) \subset W$.

Ex $T = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad W = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ 0 \end{pmatrix} \,\middle|\, z_1, z_2 \in \mathbb{C} \right\}$ is T-invariant

$U = \left\{ \begin{pmatrix} 0 \\ 0 \\ z_3 \end{pmatrix} \,\middle|\, z_3 \in \mathbb{C} \right\}$ is also T-invariant.

$X = \left\{ \begin{pmatrix} 0 \\ z_2 \\ z_3 \end{pmatrix} \,\middle|\, z_2, z_3 \in \mathbb{C} \right\}$ is not T-invariant:

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_2 \\ 3z_2 \\ 2z_3 \end{pmatrix} \notin X.$$

Ex $$T = \begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{k} & \overbrace{\quad}^{\ell} \end{array} & \\ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) & \begin{array}{l} \}\,k \\ \}\,\ell \end{array} \end{array}$$

Then $U = \mathbb{C}^k \times \{\vec{0}\}$ and $W = \{\vec{0}\} \times \mathbb{C}^\ell$ are T-invariant:

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \begin{pmatrix} z_1 \\ \vdots \\ z_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left(\begin{array}{c} A\begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \\ \hline 0 \end{array}\right) \in \mathbb{C}^k \times \{\vec{0}\}$$

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_{k+1} \\ \vdots \\ z_{k+\ell} \end{pmatrix} = \left(\begin{array}{c} 0 \\ \hline B\begin{pmatrix} z_{k+1} \\ \vdots \\ z_{k+\ell} \end{pmatrix} \end{array}\right) \in \{\vec{0}\} \times \mathbb{C}^\ell$$

Polynomials and linear maps.

Suppose $T: V \to V$ is a linear map and $p(z) = a_0 + a_1 z + \dots + a_n z^n$ is a polynomial with complex coefficients.

For each k, $0 \le k \le n$, we set $T^0 = id_V$, $T^1 = T$, $T^k = \overbrace{T \circ \dots \circ T}^{k}$

They $T^0, T^1, \dots, T^n \in Hom(V,V) = \{ S : T \to T \mid S \text{ is linear} \}$

$Hom(V,V)$ is a vector space (isomorphic to $\underset{n = \dim V}{M_{n \times n}(\mathbb{C})}$)

$\Rightarrow \forall a_0, \dots a_n \in \mathbb{C}$,

$$a_0 \underbrace{T^0}_{id_V} + a_1 T^1 + \dots + a_n T^n \in Hom(V,V).$$

We now define a map ev_T (evaluation at $T \in Hom(V,V)$)

$$ev_T : \mathbb{C}[z] \longrightarrow Hom(V,V) \quad (= \mathcal{L}(V,V))$$

$$ev_T(a_0 + a_1 z + \dots + a_n z^n) := a_0 id_V + a_1 T + a_2 T^2 + \dots + a_n T^n.$$

<u>Lemma 33.2</u> Let $T : V \to V$ be a linear map. Then

$$ev_T : \mathbb{C}[z] \longrightarrow Hom(V,V)$$

is <u>linear</u> and $\forall p, q \in \mathbb{C}[z]$

$$ev_T(p \cdot q) = ev_T(p) \cdot ev_T(q)$$

<u>Proof</u> (1) Suppose $p(z) = a_0 + a_1 z + \dots + a_n z^n$, $q(z) = b_0 + b_1 z + \dots + b_m z^m$, $\lambda, \mu \in \mathbb{C}$. Then $q(z) = b_0 + b_1 z + \dots + b_m z^m + b_{m+1} z^{m+1} + \dots + b_n z^n$ with $b_{m+1}, \dots b_n = 0$.

Now $ev_T(\lambda p + \mu q) = ev_T\left(\sum_{k=0}^{n} (\lambda a_k + \mu b_k) z^k \right)$

$$= \sum_k (\lambda a_k + \mu b_k) T^k$$

$$= \lambda \sum_k a_k T^k + \mu \sum_k b_k T^k = \lambda\, ev_T(p) + \mu\, ev_T(q).$$

$\Rightarrow ev_T$ is linear.

(2) $$p(z) \cdot q(z) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} a_i b_j \right) z^k$$

$$\Rightarrow ev_T(p(z) q(z)) = \sum_{k=0}^{n+m} \left(\sum_{i+j=k} a_i b_j \right) T^k$$

while $ev_T(p) \cdot ev_T(q) = \left(\sum_{i=0}^{n} a_i T^i \right) \left(\sum_{j=0}^{m} b_j T^j \right) = \sum_{i,j} a_i b_j T^{i+j}$

$$= \sum_{k=0}^{n+m} \left(\sum_{i+j=k} a_i b_j \right) T^k = ev_T(p \cdot q)$$