Last time Isometries, unitary maps, unitary ($U^*U = I$) and orthogonal ($A^TA = I$) matrices. 32.1

Didn't prove

Lemma 31.3 Let $A \in M_{n,n}(\mathbb{C})$. There exist a unitary matrix U s.t. UAU^{-1} is diagonal $\Leftrightarrow$ A has an orthonormal basis of eigenvectors.

Proof ($\Rightarrow$) Suppose $\exists\, U \in \mathcal{U}(n)$ s.t. $UAU^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Since the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ is orthonormal, $U \in \mathcal{U}(n)$,

$$(Ue_i, Ue_j) = (e_i, e_j) = \delta_{ij}.$$

$\Rightarrow \{Ue_1, Ue_2, \dots, Ue_n\}$ is an orthonormal basis of $\mathbb{C}^n$.

Moreover $\forall j$

$$A(Ue_j) = UDU^{-1}Ue_j = U\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} e_j\right) = U(\lambda_j e_j) = \lambda_j (Ue_j)$$

$\Rightarrow$ $Ue_1, \dots, Ue_n$ are eigenvectors of A.

($\Leftarrow$) Suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ is an orthonormal basis of $\mathbb{C}^n$ so that $\forall j, \exists \lambda_j$ with $Ab_j = \lambda_j b_j$.

Recall: $[A]_{\mathcal{BB}} = (c_{ij})$ s.t.

$$\left(Ab_j = \sum_i c_{ij} b_i \right. \qquad (= \lambda_j b_j$$

$$\Rightarrow [A]_{\mathcal{BB}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

On the other hand • $A = [A]_{\mathcal{SS}} = [I]_{\mathcal{SB}} [A]_{\mathcal{BB}} [I]_{\mathcal{BS}}$

• $[I]_{\mathcal{SB}} = (b_1 | \dots | b_n)$, which is unitary since $(b_i, b_j) = \delta_{ij}$.

• $[I]_{\mathcal{BS}} = [I]_{\mathcal{SB}}^{-1}$

$$\therefore \quad A = U\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^{-1} \quad \text{where } U = (b_1 | \dots | b_n). \qquad \square$$

Theorem (Schur) [Thm 6.14 p370 of text, Thm 1.1 p 163 of Treil]

Let V be a finite dimensional inner product space, $T: V \to V$ linear. $\exists$ an orthonormal basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of V s.t. $[T]_{\mathcal{BB}}$ is upper triangular.

<u>Proof</u>. Recall $[T]_{BB} = (c_{ij})$ if $\quad T b_j = \sum_i c_{ij} b_i$

- (c_{ij}) upper triangular $\Leftrightarrow$

$$T b_1 = c_{11} b_1$$
$$T b_2 = c_{12} b_1 + c_{22} b_2$$
$$\vdots$$
$$T b_k = c_{1k} b_1 + \cdots + c_{kk} b_k$$
$$\vdots$$

- We now use induction on $n = \dim V$.

if $\dim V = 1$, $\exists$ a basis $\{v_1\}$ of V. Then $\{b_1 = \frac{1}{\|v_1\|} v_1\}$ is orthonormal and $T b_1 = c_{11} b_1$ for some $c_{11} \in \mathbb{C}$.

Suppose theorem holds for all vector space of $\dim \leq n-1$ and $\dim V = n$.
We know T has an eigenvalue λ (the root of $p(x) = \det(xId - T)$)
Let b_1 be a corresponding eigenvector. May assume: $\|b_1\| = 1$.
Let $E = (\mathbb{C} b_1)^\perp$ Then $\dim E = \dim V - \dim(\mathbb{C} b_1) = n-1$.
Consider $T' = P_E \circ (T|_E)$ where $(T|_E)(v) = T(v)$ $\forall v \in E$
and $P_E : V \to E$ is the orthogonal projection.
By inductive assumption E has an orthonormal basis $\mathcal{B}' = \{b_2, \ldots b_n\}$ so that $[T']_{\mathcal{B}'\mathcal{B}'}$ is upper triangular:

$$T' b_2 = c_{22} b_2$$
$$T' b_3 = c_{23} b_2 + c_{33} b_3$$
$$\vdots$$
$$T' b_n = c_{2n} b_2 + \cdots + c_{nn} b_n$$

for some $(c_{ij})_{2 \leq i,j \leq n} \in M_{n-1,n-1}(\mathbb{C})$

Then $\{b_1, \ldots b_n\}$ is an orthonormal basis of V
(since $(b_1, b_1) = 1$, $(b_1, b_j) = 0$ for $j > 1$ $(b_i, b_j) = \delta_{ij}$ for $2 \leq i,j \leq n$)

and $\forall j \geq 2$ $T b_j = c_{1j} b_1 + T' b_j$ for some $c_{1j} \in \mathbb{C}$.

$\Rightarrow [T]_{BB}$ is upper triangular. $\square$

Corollary 32.1 For any $n \times n$ matrix A there exists a unitary matrix U st UAU^{-1} is upper triangular.

Proof Consider the linear map $A: \mathbb{C}^n \to \mathbb{C}^n$, $v \mapsto Av$. By Schur's thm $\exists$ a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ st $[A]_{BB}$ is upper triangular. Let $U = (b_1 | b_2 | \dots | b_n)$. Then as in proof of 31.3

$$A = U [A]_{BB} U^{-1}. \qquad \square$$

Definition $A \in M_{n,n}(\mathbb{C})$ is Hermitian if $A = A^*$.

A linear map $T: V \to V$ on an inner product space is self-adjoint (or Hermitian) if

$$T = T^*.$$

Spectral theorem

Let $T: V \to V$ be a self-adjoint linear map on a finite dim. v. space V.

(1) All eigenvalues of T are real.

(2) There exists an orthonormal basis of V consisting of eigenvectors of T.

Proof (1) Suppose λ is an eigenvalue of T. Let v be a corresponding eigenvector. Then $v \neq 0$, $Tv = \lambda v$ and $T = T^*$. Therefore

$$\lambda(v, v) = (\lambda v, v) = (Tv, v) = (v, T^* v) = (v, Tv) = (v, \lambda v) = \bar{\lambda}(v, v)$$

Since $v \neq 0$, $(v, v) > 0$. $\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

To prove (2) we need:

Lemma 32.2 (Theorem 6.10 on p 359 of text). (Compare with Lect 29 p 3)

Let $B = \{b_1, \dots, b_n\}$ be an orthonormal basis of an inner product space V, and $T: V \to V$ linear. Then

$$[T^*]_{BB} = ([T]_{BB})^*.$$

Proof $[T^*]_{BB} = (c_{ij})$ st $T^* b_j = \sum_i c_{ij} b_i$; $[T]_{BB} = (a_{ij})$ st

$T b_k = \sum_s a_{sk} b_s.$

$(T^* b_j, b_k) = (\Sigma\, c_{ij} b_i, b_k) = \Sigma\, c_{ij} (b_i, b_k) = \Sigma\, c_{ij} \delta_{ik} = c_{kj}.$

$(T^* b_j, b_k) = (b_j, T b_k) = (b_j, \sum_s a_{sk} b_s) = \Sigma\, \bar{a}_{sk} (b_j, b_s) = \Sigma\, \bar{a}_{sk} \delta_{js} = \bar{a}_{jk}.$

$\Rightarrow \quad c_{kj} = \bar{a}_{jk}$

ie $\quad [T^*]_{BB} = ([T]_{BB})^*.$

Proof of (2) of spectral theorem. By Schur's theorem there is an orthonormal basis B of V st $A = [T]_{BB}$ is upper triangular.

Then $A^* = ([T]_{BB})^* \underset{32.2}{=} [T^*]_{BB} \underset{\text{since } T = T^*}{=} [T]_{BB} = A.$

Since A is upper triangular and $A^* = (\bar{A})^T$ is upper triangular, A is diagonal and $A = \bar{A}$.

Since $[T]_{BB}$ = diagonal matrix, $T b_j = \lambda_j b_j \quad \forall b_j \in B$ $\quad \square$