Last time 1) Given $A: \mathbb{C}^n \to \mathbb{C}^m$, $y \in \mathbb{C}^m$

$\|Ax - y\|^2$ is minimal $\iff$ $A^*Ax = A^*y$

2) if $N(A) = 0$ then A^*A is invertible and

$$A^*A x = A^*y$$

has a unique solution (which minimizes $\|Ax-y\|^2$)

3) V/inner product space, $E \subseteq V$ subspace Then

$$E \subseteq (E^\perp)^\perp$$

if $\dim V < \infty$, $E = (E^\perp)^\perp$.

4) $T: V \to W$ linear map between inner product spaces. Then

(0) $(T^*)^* = T$

(1) $N(T^*) = R(T)^\perp$. Hence $N(T^*)^\perp = R(T)$

(2) $N(T) = R(T^*)^\perp$. Hence $N(T)^\perp = R(T^*)$

Definition A linear map $T: V \to W$ between two inner product spaces is an <u>isometry</u> if

$$\|Tv\| = \|v\| \quad \forall v \in V.$$

Remarks (1) if $(Tv_1, Tv_2) = (v_1, v_2)$ $\forall v_1, v_2 \in V$,

then $\|Tv\|^2 = (Tv, Tv) = (v, v) = \|v\|^2$

so T is an isometry.

Polarization identity (HW#11 problem 6) $\Rightarrow$

If T is an isometry then $(Tv_1, Tv_2) = (v_1, v_2)$ $\forall v_1, v_2 \in V$.

(2) If T is an isometry then $N(T) = 0$. This is because:

if $T(v) = 0$ then $\|v\| = \|Tv\| = \|0\| = 0$. $\Rightarrow v = 0$.

(3) If $T: V \to W$ is an isometry and $\dim W < \infty$

then $\dim V \le \dim W$.

This is because $\dim W \ge \dim R(T) = \dim V - \dim N(T)$

$= \dim V - 0.$

Def A map $T: V \to W$ between two Hermitian vector spaces is *unitary* if it is a surjective isometry.

Note 1) Any unitary map is automatically invertible.

2) Any isometry $T: V \to V$ is automatically unitary.

Lemma 31.1 A linear map $T: V \to W$ between two Hermitian vector spaces is an isometry $\Leftrightarrow$ $T^* \circ T = Id_V$.

Proof ($\Rightarrow$) If T is an isometry then $\forall v_1, v_2 \in V$

$$(v_1, v_2) = (Tv_1, Tv_2) = (v_1, T^* T v_2)$$

$$\Leftrightarrow \quad v_2 = T^* T v_2 \quad \forall v_2$$

$$\Leftrightarrow \quad Id_V = T^* T.$$

($\Leftarrow$) If $T^* T = Id_V$, then $(v_1, v_2) = (v_1, T^* T v_2)$ $\forall v_1, v_2 \in V$

$$\Rightarrow (v_1, v_2) = (Tv_1, Tv_2)$$

$\Rightarrow$ T is an isometry. $\square$

Def An $n \times n$ complex matrix U is *unitary* if the corresponding map $U: \mathbb{C}^n \to \mathbb{C}^n$ is an isometry, $\forall z, w \in \mathbb{C}^n$

$$(Uz, Uw) = (z, w)$$

(i.e. $w^* z = (Uw)^* Uz = w^* U^* U z$)

Thus U is unitary $\Leftrightarrow$ $U^* U = I$.

$$U(n) := \{ U \in M_{n,n}(\mathbb{C}) \mid U^* U = I \}$$

Homework $\Rightarrow$ $\forall n$ $U(n)$ is a group.

Moreover, $U \in U(n) \Leftrightarrow$ U is invertible and $U^{-1} = U^*$

$\Leftrightarrow$ columns of U are orthonormal basis of $\mathbb{C}^n$

Def A *real* matrix $O \in M_{n,n}(\mathbb{R})$ is *orthogonal* if $O^T O = I$

Note $O \in M_{n,n}(\mathbb{R})$ is orthogonal $\Leftrightarrow$ $O \in M_{n,n}(\mathbb{R}) \cap U(n)$

$\Leftrightarrow$ O is invertible and $O^{-1} = O^T$

$\Leftrightarrow$ columns of O are orthonormal basis of $\mathbb{R}^n$.

$O(n) = \{ A \in M_{n,n}(\mathbb{R}) \mid A^T A = I \}$
$O(n)$ is also a group.

Lemma 31.2 (Treil, prop 6.4 p148)
(1) If $U \in U(n)$ then $|\det(U)| = 1$
(2) if $A \in O(n)$ then $\det O = \pm 1$
(3) If $U \in U(n)$ / $Uv = \lambda v$ for some $v \neq 0$, Then $|\lambda| = 1$.

Proof (1,2) $\forall\, U \in M_{n,n}(\mathbb{C})$
$$\det(U^*) = \det((\overline{U})^T) = \det(\overline{U}) = \overline{(\det U)}. \quad \text{HW 12 \#8}$$

So if $U^* U = I$, then
$$1 = \det I = \det U^* U = \det(U^*) \det U = \overline{(\det U)} \det U = |\det(U)|^2$$
$\Rightarrow |\det U| = 1.$

If $A \in O(n) \subseteq U(n)$, then $\det A \in \mathbb{R}$.
So $|\det A| = 1 \Rightarrow \det A = \pm 1.$

(3) $Uv = \lambda v$ for $v \neq 0 \Rightarrow \|\lambda v\|^2 = \|U(v)\|^2 = \|v\|^2$
On the other hand $\|\lambda v\|^2 = (\lambda v, \lambda v) = \lambda \overline{\lambda} (v, v) = |\lambda|^2 \|v\|^2$.
$\Rightarrow |\lambda|^2 \|v\|^2 = \|v\|^2$.
Since $v \neq 0$, $\|v\|^2 \neq 0 \Rightarrow |\lambda|^2 = 1$. □

Recall Two $n \times n$ matrices A & B are similar (or conjugate) if $\exists$ invertible $n \times n$ matrix S s.t. $A = SBS^{-1}$.

Definition Two $n \times n$ matrices A & B are unitarily equivalent if $\exists\, U \in U(n)$ s.t. $A = UBU^{-1}$ $(= UBU^*)$.

Lemma 31.3 An $n \times n$ complex matrix is unitarily equivalent to a diagonal matrix $\Leftrightarrow$ the matrix has an orthonormal basis of eigenvectors.

<u>Proof</u> Suppose $A \in M_{n,n}(\mathbb{C})$, $U \in \mathcal{U}(n)$, $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ and
$A = UDU^{-1}$.

Consider the standard basis $\{e_1 \dots e_n\}$ of $\mathbb{C}^n$. It's orthonormal.
$(Ue_i, Ue_j) = (e_i, e_j) = \delta_{ij}$. $\Rightarrow \{Ue_1, \dots, Ue_n\}$ is orthonormal, hence a basis of $\mathbb{C}^n$. Moreover

$$A(Ue_j) = UDU^{-1}Ue_j = U(De_j) = U(d_j e_j) = d_j \cdot Ue_j$$

$\{Ue_1, \dots, Ue_n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

$(\Leftarrow)$ Suppose $\mathcal{B} = \{b_1 \dots b_n\}$ is an orthonormal basis of $\mathbb{C}^n$ with $Ab_j = d_j b_j$

Then $[A]_{\mathcal{B}\mathcal{B}} [b_j]_{\mathcal{B}} = [Ab_j]_{\mathcal{B}} = [d_j b_j]_{\mathcal{B}} = d_j [b_j]_{\mathcal{B}}$

But $[b_j]_{\mathcal{B}} = e_j$. So

$$[A]_{\mathcal{B}\mathcal{B}} e_j = d_j e_j$$

$$\Rightarrow [A]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

On the other hand $A = [A]_{\mathcal{S}\mathcal{S}} = [I]_{\mathcal{S}\mathcal{B}} [A]_{\mathcal{B}\mathcal{B}} [I]_{\mathcal{B}\mathcal{S}}$

$[I]_{\mathcal{B}\mathcal{S}} = (b_1 | b_2 | \dots | b_n) =: B$ which is <u>unitary</u>:

$\delta_{ij} = (b_i, b_j) = b_j^* b_i \Rightarrow B^* B = I$. And $[I]_{\mathcal{S}\mathcal{B}} = [I]_{\mathcal{B}\mathcal{S}}^{-1}$

$$\therefore \quad A = B^{-1} \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} B = B^{-1} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} (B^{-1})^{-1}.$$

and B^{-1} is also unitary $\square$