<u>Last time</u> 1) Given a linear map $T: V \to W$ between two inner product spaces $\exists$ unique linear map $T^*: W \to V$ s.t.

$$(Tv, w) = (v, T^* w) \quad \forall v \in V, w \in W$$

T^* = the adjoint of T

2) if $V = \mathbb{C}^n$, $W = \mathbb{C}^m$ and $T = (a_{ij})$ Then

$(T^*)_{ij} = \bar{a}_{ji}$, ie the matrix of T^* is the conjugate transpose of the matrix of T.

<u>Ex</u>

$$A = \begin{pmatrix} i & 2i+1 \\ 0 & 3+i \end{pmatrix} \qquad A^* = \begin{pmatrix} \bar{i} & \bar{0} \\ \overline{2i+1} & \overline{3+i} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 1-2i & 3-i \end{pmatrix}$$

<u>Ex</u> $A = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$ $\quad A^* = (\bar{z}_1, \dots \bar{z}_n)$.

and $(z, w) = \sum z_j \bar{w}_j = (\bar{w}_1 \dots \bar{w}_n) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = w^* z$.

Hence $\forall A \in M_{m,n}(\mathbb{C})$ $z \in \mathbb{C}^n$, $w \in \mathbb{C}^m$

$$(z, A^* w) = (A^* w)^* z = w^* (A^*)^* z = w^* A z = (Az, w).$$

<u>Remark</u> Similar results hold for real vector spaces: given an $\mathbb{R}$-linear map $T: V \to W$ between real inner product spaces, $\exists$ unique linear map $T^*: W \to V$ st

$$(Tv, w) = (v, T^* w) \quad \forall v \in V, w \in W$$

Also if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ then $(T^*)_{ij} = T_{ji}$

ie. $A^* = A^T$.

<u>Least squares</u> (p 361 in text)

We are given real data $(t_1, y_1), (t_2, y_2) \dots (t_N, y_N) \in \mathbb{R}^2$.

<u>Want:</u> Find $c, d \in \mathbb{R}$ so that

$$\sum_{i=1}^{\tilde{N}} (y_i - f(t_i))^2$$

is <u>minimal</u> where $f(t) = ct + d$.

What does it have to do with linear maps?

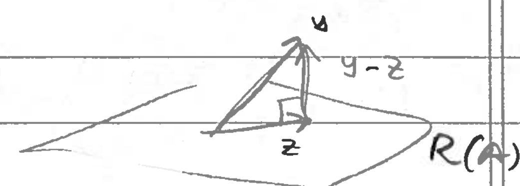
Let $A = \begin{pmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_N & 1 \end{pmatrix} \in M_{N,2}(\mathbb{R})$, $x = \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2$.

Then $Ax = \begin{pmatrix} t_1 c + d \\ t_2 c + d \\ \vdots \\ t_N c + d \end{pmatrix} \Rightarrow \sum_{i=1}^{N} (y_i - f(t_i))^2 = \sum (y_i - (Ax)_i)^2$

$= \| y - Ax \|^2$

The least squares fit problem becomes: given $A \in M_{N,2}(\mathbb{R})$, $y \in \mathbb{R}^N$ find $x \in \mathbb{R}^2$ so that $\|Ax - y\|^2$ is minimal.

<u>Note</u> $Ax \in R(A)$. And $\forall z \in R(A)$ $\|z - y\|^2$ is minimal $\Leftrightarrow$ $z - y \perp R(A)$



We now <u>generalize</u> the problem: given $A: \mathbb{C}^n \to \mathbb{C}^m$, $y \in \mathbb{C}^m$ find $x \in \mathbb{C}^n$ so that $Ax - y \perp R(A)$.

Now if $A = (a_1 | \dots | a_n)$ $R(A) = \operatorname{span}\{a_1, \dots, a_n\}$.

So $Ax - y \perp R(A) \Leftrightarrow Ax - y \perp a_j \quad \forall j = 1, \dots, n$.

$\Leftrightarrow \quad 0 = (Ax - y, a_j) = a_j^* (Ax - y) \quad \forall j$

$$\Leftrightarrow \begin{pmatrix} a_1^* \\ \hline a_2^* \\ \hline \vdots \\ a_n^* \end{pmatrix} (Ax - y) = 0 \Leftrightarrow A^*(Ax - y) = 0$$

$$\Leftrightarrow A^* A x = A^* y$$

<u>We proved</u>:

<u>Lemma 30.1</u> $\|Ax - y\|^2$ is minimal $\Leftrightarrow$ $A^* A x = A^* y$

<u>Lemma 30.2</u> Suppose $A \in M_{m,n}(\mathbb{C})$, with $N(A) = 0$. Then the equation

$(\star)$ $A^* A x = A^* y$

has a unique solution: $x_0 = (A^* A)^{-1} A^* y$.

<u>We first prove</u>: <u>Lemma 30.3</u> $\forall A \in M_{m,n}(\mathbb{C})$

$N(A^* A) = N(A)$.

Proof $v \in N(A) \Rightarrow Av = 0 \Rightarrow A^* A v = 0 \Rightarrow v \in N(A^* A)$

$v \in N(A^* A) \Rightarrow A^* A v = 0 \Rightarrow (A^* A v, v) = 0 \Rightarrow (Av, Av) = 0 \Rightarrow Av = 0$

$\Rightarrow \quad v \in N(A).$ $\square$

Proof of Lemma 30.2 Suppose $N(A) = 0$. Then by 30.3 $N(A^*A) = 0$. But A^*A is a square matrix so $N(A^*A) = 0 \Rightarrow A^*A$ is invertible. $\Rightarrow (A^*A)^{-1}$ exists.

$\Rightarrow x_0 = (A^*A)^{-1} A^*y$ is the (unique) solution of $A^*Ax = A^*y$. $\square$

Back to least squares problem: Since $t_1 \ldots t_N$ are distinct $\begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ are lin. independent $\Rightarrow N\left(\begin{pmatrix} t_1 & 1 \\ \vdots & \vdots \\ t_N & 1 \end{pmatrix}\right) = 0$.

$\Rightarrow$ $A^*A\,x = A^*y$ has a unique solution:

$$x_0 = (A^*A)^{-1}(A^*y)$$

For this x_0,

$$Ax_0 - y \perp R(A)$$

and so $\|Ax_0 - y\|$ is minimal.

Remark Let V be an inner product space, $E \leq V$ a subspace. Then $E \leq (E^\perp)^\perp$: $\forall x \in E$, $\forall y \in E^\perp$ $(x, y) = 0$

$\Rightarrow x \in (E^\perp)^\perp \Rightarrow E \leq (E^\perp)^\perp$.

If $\dim V < \infty$, $\dim E + \dim E^\perp = \dim V$

$\dim E^\perp + \dim((E^\perp)^\perp) = \dim V$

$\Rightarrow \dim(E^\perp)^\perp = \dim E$. Since $E \leq (E^\perp)^\perp$, $E = (E^\perp)^\perp$.

Lemma 30.4 Let $T: V \to W$ be a linear map between two finite dimensional inner product spaces. Then

(1) $N(T^*) = R(T)^\perp$. Hence $N(T^*)^\perp = (R(T)^\perp)^\perp = R(T)$.

(2) $N(T) = R(T^*)^\perp$ Hence $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$.

Proof Note first that $(T^*)^* = T$. This is because $\forall w \in W, \forall v \in V$

$$(w, Tv) = (T^*w, v) = (w, (T^*)^*v).$$

$$\Rightarrow Tv = (T^*)^*v \quad \forall v \quad \Rightarrow \quad T = (T^*)^*.$$

Therefore $(1) \Rightarrow (2)$.

Now

$$w \in N(T^*) \Leftrightarrow T^*w = 0 \Leftrightarrow (T^*w, v) = 0 \quad \forall v \in V$$
$$\Leftrightarrow (w, Tv) = 0 \quad \forall v \in V$$
$$\Leftrightarrow w \perp R(T).$$
$$\therefore \quad N(T^*) = R(T)^\perp.$$

Definition A linear map $T: V \to W$ between two inner product spaces is an isometry if

$$(Tv_1, Tv_2) = (v_1, v_2) \qquad \forall v_1, v_2 \in V.$$

Remarks 1) if T is an isometry, then T is injective:

if $T(v) = 0$, then $(v, v) = (T(v), T(v)) = (0, 0) = 0$

$\Rightarrow v = 0. \rightarrow T$ is injective

2) if T is an isometry, then $\forall v \in V$

$$\|T(v)\|^2 = (Tv, Tv) = (v, v) = \|v\|^2.$$

$\rightarrow T$ preserves $\|\cdot\|$.

Note Polarization identity (Problem 6 on HW11) $\Rightarrow$

if $\|Tv\| = \|v\| \ \forall v$ then $(Tv_1, Tv_2) = (v_1, v_2) \quad \forall v_1, v_2 \in V$.

3) If $T: V \to W$ is an isometry, then $\dim W \geq R(T) = \dim V - N(T)$
$= \dim V$.

Def A map $T: V \to W$ is unitary if it is a surjective isometry (and then T automatically an isomorphism).