Last time: • Defined a vector space over $\mathbb{R}$ as a (nonempty) set V together with two operations $\cdot : \mathbb{R}\times V \to V$ (scalar multiplication) and $+ : V\times V \to V$ (vector addition) and $\vec{0} \in V$ subject to 8 conditions: $+$ is commutative and associative, there are additive inverses, $1\cdot\vec{v} = \vec{v}$ for all $\vec{v} \in V$, ...

- defined vector subspaces: a subspace W of a vector space V is a (nonempty) subset of V which is preserved under vector addition and scalar multiplication of V. A vector subspace then is a vector space over $\mathbb{R}$
- defined linear maps between vector spaces
- proved a few easy lemmas.

Today: linear combinations, span, linear independence

First

Example/notation Fix $n, m > 0$ ($n, m \in \mathbb{N}$)

$M_{m,n}(\mathbb{R})$ = the set of $m\times n$ matrices with real coefficients

$$= \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \middle| \; a_{ij} \in \mathbb{R} \right\}$$

$M_{m,n}(\mathbb{R})$ is a vector space over $\mathbb{R}$

$$\lambda \cdot (a_{ij}) = (\lambda a_{ij})$$

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

Definition Let V be a vector space, $k > 0$,

A linear combination of $\vec{v}_1, \dots \vec{v}_k \in V$ is the vector

$$\vec{u} = \alpha_1 \vec{v}_1 + \dots + \alpha_k \vec{v}_k$$

for some $\alpha_1, \dots \alpha_k \in \mathbb{R}$.

Ex $V = \mathbb{R}^2$ $\quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Any vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a linear combination of $\vec{e}_1, \vec{e}_2$ since

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 .$$

Ex $V = \mathcal{P}_3 = \{ p(x) \in \mathbb{R}[x] \mid \deg p \leq 3 \}$
$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \}$.

A linear combination of $\vec{v}_1 = x$ and $\vec{v}_2 = x^3$ is a polynomial $\alpha_1 x + \alpha_2 x^3$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$.

Definition A set of vectors $\{v_1 .. v_m\}$ in a vector space V spans V if $\forall \vec{u} \in V$ $\exists \alpha_1 .. \alpha_m \in \mathbb{R}$ s.t.

$$\vec{u} = \alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m.$$

Ex $\{ e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix} \}$ spans $\mathbb{R}^2$ since

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1 e_1 + x_2 e_2$$

$\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \}$ also spans $\mathbb{R}^2$. $\{ \begin{pmatrix}1\\0\end{pmatrix} \}$ doesn't span $\mathbb{R}^2$: $\nexists\, \alpha \in \mathbb{R}$ s.t. $\begin{pmatrix}0\\1\end{pmatrix} = \alpha \begin{pmatrix}1\\0\end{pmatrix}$.

Definition A set $\{\vec{v}_1, .. \vec{v}_m\} \subseteq V$ (V = a vector space) is linearly dependent if $\exists \alpha_1 .. \alpha_m \in \mathbb{R}$ not all zero s.t.

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_m \vec{v}_m = \vec{0}$$

Ex $\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \} \subseteq \mathbb{R}^2$ is linearly dependent:

$$1 \cdot \begin{pmatrix}1\\0\end{pmatrix} + (-1) \begin{pmatrix}1\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Def A subset $\{\vec{v}_1 .. \vec{v}_m\}$ of a vector space V is linearly independent if it's not linearly dependent

$$\alpha_1 \vec{v}_1 + \dots + \alpha_m \vec{v}_m = \vec{0} \implies \alpha_1 = \alpha_2 \dots = \alpha_m = 0.$$

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ is linearly independent.

Check

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha_1 1 + \alpha_2 1 \\ \alpha_1 \cdot 0 + \alpha_2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 + 0 = 0.$$

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$ is linearly independent

Check

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = 0.$$

Ex $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq M_{2,2}(\mathbb{R})$

is linearly independent and spans $M_{2,2}(\mathbb{R})$.

Lemma 3.1 Suppose $\{v_1, \dots v_m\} \subseteq V$ is linearly independent. Then

$$\sum_{i=1}^{m} \alpha_i v_i = \sum_{i=1}^{m} \beta_i v_i \Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots \alpha_m = \beta_m.$$

Proof if $\alpha_1 v_1 + \dots + \alpha_m v_m = \beta_1 v_1 + \dots + \beta_m v_m$

then $(\alpha_1 v_1 - \beta_1 v_1) + \dots + (\alpha_m v_m - \beta_m v_m) = \vec{0}$

$\Rightarrow (\alpha_1 - \beta_1)\vec{v}_1 + \dots + (\alpha_m - \beta_m) v_m = \vec{0}$

Since $\{v_1 \dots v_m\}$ is linearly independent $\alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0 \dots$

$\alpha_m - \beta_m = 0$. D

Definition A subset $\{v_1 \dots v_m\}$ of a vector space V is a basis if it is linearly independent and spans V.

Note If $\{v_1 \dots v_m\}$ is a basis of V then $\forall\, \vec{u} \in V$ $\exists\, \alpha_1 \dots \alpha_m \in \mathbb{R}$ s.t. $u = \alpha_1 v_1 + \dots + \alpha_m v_m$ (since $\{v_1 \dots v_m\}$ spans V)

Moreover $\alpha_1, \ldots, \alpha_m$ are unique: if

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = u = \beta_1 v_1 + \cdots + \beta_m v_m$$

Then by lemma 3.1 $\alpha_1 = \beta_1, \ldots \alpha_m = \beta_m$.

Thus if $\{v_1, \ldots v_m\}$ is a basis of V then $\forall u \in V$ $\exists$ unique $\alpha_1(u), \ldots \alpha_m(u) \in \mathbb{R}$ (α's depend on u!) st

$$u = \sum_{i=1}^{m} \alpha_i(u)\, v_i$$

In other words the map

$$\mathbb{R}^m \longrightarrow V, \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \longmapsto \alpha_1 v_1 + \cdots + \alpha_m v_m$$

is a <u>bijection</u>.

<u>Ex</u> $V = \mathbb{R}^2$ $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is also a basis: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 - x_2 \\ 0 \end{pmatrix}$

$$= x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x_1 - x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ spance $\mathbb{R}^2$ and we've checked linear independence.

<u>Ex</u> $\{1, x, \ldots x^n\}$ is a basis of $P_n = \{p \in \mathbb{R}[x] \mid \deg p \le n\}$

<u>Def</u> A vector space V is <u>finite dimensional</u> if $\exists \{v_1 \ldots v_m\} \subseteq V$ st. $\{v_1 \ldots v_m\}$ is a basis.

<u>Key theorems</u> 1) Suppose V is a vector space and there is a set $\{v_1 \ldots v_m\} \subseteq V$ which spans V. Then V has a finite basis.

2) Suppose V is a finite dimensional vector space and $\{v_1 \ldots v_m\}$, $\{u_1, \ldots u_n\}$ are two bases of V. Then $m = n$.

We'll prove them next week...