Last time:

Gram-Schmidt: If V is an inner product space then every finite dimensional subspace E of V has an orthogonal basis, $\{v_1 \ldots v_n\}$ (ie. $(v_j, v_i) = 0$ if $i \neq j$, $(v_i, v_i) \neq 0$)

Hence it has an orthonormal basis: Let $b_i = \frac{1}{\|v_i\|} v_i$.

Then $(b_i, b_j) = 0$ for $i \neq j$ $(b_i, b_i) = \frac{1}{\|v_i\|^2}(v_i, v_i) = 1$.

Construction (secretely) uses orthogonal projections:

recall: P is a projection if $P^2 = P$.

P is an orthogonal projection if additionally $R(P) \perp N(P)$.

We proved: • If V is finite dimensional, $E \subseteq V$ subspace then

$$V = E \oplus E^\perp$$

• $\forall$ subspace $E \subseteq V$ $\exists$ $P_E : V \to V$ st $R(P_E) = E$ $N(P_E) = E^\perp$ and $P_E^2 = P_E$.

Note: $P_E|_{E^\perp} = 0$ since $E^\perp = N(P_E)$

$P_E|_E = id_E$. Reason: $w \in E = P_E(V) \Rightarrow w = P_E(v)$ for some $v \in V$. $\Rightarrow$ $P_E(w) = \underbrace{P_E(P_E(v))}_{} = P_E(v) = w$

since $P_E^2 = P_E$.

Note $P_E : V \to V$ is the unique map such that

$P_E|_E = id_E$, $P_E|_{E^\perp} = 0$:

if $T: V \to V$ is any other map with this property then $\forall v \in V$, $\exists u \in E$, $w \in E^\perp$ st $v = u + w$.

And then $T(v) = T(u) + T(w) = u + 0$

$P_E(v) = P_E(u) + P_E(w) = u + 0$

$\Rightarrow$ $T = P_E$.

Remark if $P_E : V \to V$ is an orthogonal projection with $R(P_E) = E$ then $Id_V - P_E : V \to V$ is also an orthogonal projection, but onto $E^\perp$: $\forall$ $v \in E^\perp$ $(Id_V - P_E)(v) = v - P_E(v)$

$= v - 0 = v.$ $\quad \forall\, u \in E \quad (\mathrm{id}_V - P_E)(u) = u - P_E(u) = u - u = 0.$

Remark

If $\{b_1, \dots, b_n\}$ is an orthonormal basis of E then $\forall v \in V$

$P_E(v) = \sum_{i=1}^{n} (v, b_i)\, b_i$. [In Dirac notation: $P_E = \sum_{i=1}^{n} |i\rangle\langle i|$]

Check

if $v \in E^\perp$ $\sum (v, b_i)\, b_i = \sum 0 \cdot b_i = 0$

if $v \in E$, $v = \sum a_j b_j$ for some $a_i \in \mathbb{C}$.

$\Rightarrow \quad \sum (v, b_i)\, b_i = \sum_i \left(\sum a_j b_j, b_i\right) b_i = \sum a_i (b_i, b_i)\, b_i = \sum a_i b_i = v.$

Remark $P_E(v) =$ the vector in E so that $\|v - P_E(v)\|$ is minimal.

Proof Recall: if $x \perp y$ $\quad \|x+y\|^2 = (x+y, x+y) = \|x\|^2 + (x,y) + (y,x) + \|y\|^2$
$= \|x\|^2 + \|y\|^2.$

$\Rightarrow \forall\, w \in E$

$$\|v - w\|^2 = \|\underbrace{v - P_E(v)}_{\in E^\perp} + \underbrace{P_E(v) - w}_{\in E}\| = \|v - P_E(v)\|^2 + \|P_E(v) - w\|^2 \geq \|v - P_E(v)\|^2.$$

remember: $\mathrm{id} - P_E$ projects onto $E^\perp$!

Adjoint operators and adjoint matrices

Theorem 29.1 Let $T: V \to W$ be a linear map between two vector spaces with inner products. There exists a unique linear map $T^*: W \longrightarrow V$ (the adjoint of T) so that

$$(T^* w, v) = (w, Tv) \quad \forall\, w \in W,\ v \in V$$

(Note the unfortunate clash of notation with $T^*: W^* \longrightarrow V^*$).
Let's change this T^* to $T^\vee$. Thus $\forall\, \ell: W \to \mathbb{C}$, $T^\vee(\ell) := \ell \circ T: V \to \mathbb{C}$

Proof (uniqueness) Suppose $S_1, S_2 : W \to V$ two linear maps so that
$$(S_1 w, v) = (w, Tv) \quad \forall v \in V$$
$$(S_2 w, v) = (w, Tv) \quad \forall w \in W$$

Then $0 = (w, Tv) - (w, Tv) = (S_1 w, v) - (S_2 w, v) = (S_1 w - S_2 w, v)$

Since v is arbitrary $S_1 w - S_2 w = 0 \quad \forall w \in W$

Since w is arbitrary, $S_1 - S_2 = 0$.

(existence) Define $S : W \to V$ by $S = (\#)^{-1} \circ T^\vee \circ \#$.

$$\begin{tikzcd} W \arrow[r, dashed, "S"] \arrow[d, "\#"'] & V \\ W^* \arrow[r, "T^\vee"'] & V^* \arrow[u, "(\#)^{-1}"'] \end{tikzcd}$$

Note $\#, (\#)^{-1}$ are antilinear so S is $\mathbb{C}$-linear.

We check that $\forall w \in W, \forall v \in V$
$$(Sw, v) = (w, Tv).$$

This amounts to unravelling definitions: $S = (\#)^{-1} \circ T^\vee \circ \#$

$\iff \quad \# \circ S = T^\vee \circ \#$

$\iff \quad \#(Sw) = T^\vee(\# w) \quad \forall w \in W$

$\iff \forall w \in W, \forall v \in V \quad (\# S(w))(v) = (T^\vee(\# w))(v)$

$\iff$ —//— $\quad (S(w), v) = (\# w)(Tv) = (w, Tv)$ ☐

We now rename S as T^*. Thus $T^* = (\#)^{-1} \circ T^\vee \circ \#$

Special case: $V = \mathbb{C}^n$, $W = \mathbb{C}^m$ $\quad T(v) = (a_{ij}) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $T : \mathbb{C}^n \to \mathbb{C}^m$

$\mathbb{C}^n, \mathbb{C}^m$ have standard inner products. Then
$$T^* e_j = \sum_{i=1}^{m} (T^*)_{ij} e_i$$

— On the one hand.
$$(e_j, Te_k) = (T^* e_j, e_k) = (\sum_i (T^*)_{ij} e_i, e_k)$$
$$= \sum_i (T^*)_{ij} \delta_{ik} = (T^*)_{kj}$$

On the other hand
$$(e_j, Te_k) = (e_j, \sum_i a_{ik} e_i) = \sum_{i=1} \overline{a}_{ik} \delta_{ji} = \overline{a}_{jk}$$

$$\therefore \quad (T^*)_{kj} = \overline{a}_{jk}$$

ie The matrix of T^* is the conjugate transpose of T.

Def The adjoint matrix A^* of a matrix A is the conjugate transpose:

$$A^* = (\bar{A})^T$$

Ex $\begin{pmatrix} 1 & i \\ 2i & 0 \end{pmatrix}^* = \begin{pmatrix} \bar{1} & \overline{2i} \\ \bar{i} & \bar{0} \end{pmatrix} = \begin{pmatrix} 1 & -2i \\ -i & 0 \end{pmatrix}$.

Remark $\forall z, w \in \mathbb{C}^n$

$$(z, w) = \Sigma z_j \bar{w}_j = (\bar{w}_1 \ldots \bar{w}_n)\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = w^* z.$$

Remark $\forall$ matrices A, B

$$(AB)^* = (\bar{A}\bar{B})^T = \bar{B}^T \bar{A}^T = B^* A^*$$

So $\forall z, w$

$$(A^* z, w) = w^* A^* z = (Aw)^* z = (z, Aw).$$

Note If $A \in M_{m,n}(\mathbb{C})$, $A^* \in M_{n,m}(\mathbb{C})$.

$\Rightarrow$ $A^* A$ is $n \times n$ matrix.

Lemma 29.2 $N(A^* A) = N(A)$

Proof if $Av = \vec{0}$, then $A^* A v = A^* \vec{0} = \vec{0}$. $\Rightarrow N(A) \subseteq N(A^* A)$

Suppose $(A^* A)(v) = \vec{0}$. Then

$$0 = v^* \vec{0} = v^* A^* A v = (Av, Av) \Rightarrow A(v) = 0$$

$\Rightarrow$ $v \in N(A)$.