04/8/2019
Mo
28.1

Last time · triangle inequality for $\|\cdot\|$, $\|x\| = (x,x)^{1/2}$.

If $\{v_1 \ldots v_n\} \subset V$ $(v_i, v_j) = 0$ for $i \neq j$, $(v_i, v_i) \neq 0$

Then $\{v_1 \ldots v_n\}$ is lin. independent.

Gram-Schmidt : Any finite dimensional inner product space has an orthogonal basis.

More precisely we prove:

(Gram-Schmidt algorithm)

Lemma 28.1 (compare with Thm 6.4 of Friedberg et al.)

Let V be an inner product space, $E \subseteq V$ a subspace, $\{w_1, \ldots, w_n\}$ a basis of E. Define $\{v_1 \ldots v_n\} \subseteq E$ recursively by

$$v_1 := w_1$$

$$v_2 := w_2 - \frac{(w_2, v_1)}{\|v_1\|^2} v_1$$

$$\vdots$$

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{(w_k, v_j)}{\|v_j\|^2} v_j$$

$$\vdots$$

Then $\{v_1 \ldots v_n\}$ is a set of orthogonal nonzero vectors in E hence a basis of E.

Example $\{w_1 = (1,0,1,0)^T,\ w_2 = (1,1,1,1)^T,\ w_3 = (0,1,2,1)^T\} \subseteq \mathbb{C}^4$

$$v_1 = (1,0,1,0)^T$$

$$v_2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{((1,1,1,1)^T, (1,0,1,0)^T)}{\|(1,0,1,0)^T\|^2} \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$v_3 = \begin{pmatrix}0\\1\\2\\1\end{pmatrix} - \frac{((0,1,2,1),(1,0,1,0))}{2} \cdot \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{((0,1,2,1),(0,1,0,1))}{2} \cdot \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\2\\1\end{pmatrix} - \frac{0\cdot1+1\cdot0+2\cdot1+0}{2} \cdot \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{0\cdot0+1\cdot1+2\cdot0+1\cdot1}{2}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0-1\\1-0-1\\2-1-0\\1-0-1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$$

Proof Induction on $n = \dim E$. Base case: $n=1$. Then $v_1 = w_1$. ✓

Suppose claim holds for all subspaces $E' \subseteq V$ with $\dim E' = n-1$. In particular it holds for $E' = \text{span}\{w_1, \dots w_{n-1}\}$.

By assumption we have a set of nonzero orthogonal vectors $\{v_1, \dots v_{n-1}\} \subseteq E' = \text{span}\{w_1 \dots w_{n-1}\}$.

Let $v_n = w_n - \sum_{j=1}^{n-1} \frac{(w_n, v_j)}{\|v_j\|^2} v_j$.

Since $w_n \notin \text{span}\{w_1 \dots w_{n-1}\}$, $v_n \neq 0$.

<u>Remains to check:</u> $\forall k < n$, $(v_n, v_k) = 0$. Now

$$(v_n, v_k) = \left(w_n - \sum \frac{(w_n, v_j)}{\|v_j\|^2} v_j, v_k\right) = (w_n, v_k) - \sum_{j=1}^{n-1} \frac{(w_n, v_j)}{\|v_j\|^2}(v_j, v_k)$$

$$= (w_n, v_k) - \frac{(w_n, v_k)}{\|v_k\|^2}(v_k, v_k) = 0.$$ $\square$

<u>Orthogonal projections</u>

<u>Recall</u>. A linear map $P: V \to V$ is a <u>projection</u> $\Leftrightarrow$ $P \circ P = P$

- V is a direct sum of the subspaces $U, W \subseteq V$ if $\forall v \in V$ $\exists$ unique $u = u(v) \in U$, $w = w(v) \in W$ s.t.

  $v = u + w$.

  This is equivalent to: (i) $V = U + W$ and (ii) $U \cap W = \{0\}$.

  It's also equivalent to

  $$U \times W \to V, \quad (u, w) \mapsto u + w$$

  is an isomorphism of vector spaces

<u>You proved:</u> $P: V \to V$ is a projection $\Leftrightarrow$ $V = N(P) \oplus R(P)$

<u>Solution</u> ($\Rightarrow$) $\forall v \in V$, $v = (v - P(v)) + P(v)$ $\quad P(v) \in R(P)$

$P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0 \Rightarrow v - P(v) \in N(P)$

$\Rightarrow V = N(P) + R(P)$. Moreover

$v \in N(P) \cap R(P) \Leftrightarrow P(v) = 0$ and $v = P(w)$ for some $w \in V$

But then $0 = P(v) = P(P(w)) = P(w)$ (since $P^2 = P$) $= v$.

So $N(P) \cap R(P) = \{0\}$. $\square$

($\Leftarrow$) If $V = E \oplus F$, $\forall v \in V\ \exists!\ w \in E, u \in F$ st $v = w + u$. Set $P(v) = w$.

Definition Let V be an inner product space, $U, W \subseteq V$ subspaces.

U is orthogonal to W $(U \perp W) \Leftrightarrow$

$\forall u \in U, \forall w \in W \quad (u, w) = 0 \quad$ (ie $u \perp w$).

A projection $P : V \to V$ is orthogonal if

$$N(P) \perp R(P).$$

Ex V inner product space, $w \in V$, $\|w\| = 1$, $E = \mathbb{C}w$

$$P_E : V \to V, \quad P_E(v) = (v, w)\, w$$

is an orthogonal projection onto E.

Check $P_E(P_E(v)) = P_E((v,w)\,w) = ((v,w)\,w, w)\, w$

$$= (v,w) \cdot \underbrace{(w,w)}_{= \|w\|^2 = 1} \cdot w = (v,w)\, w = P_E(v).$$

$\Rightarrow P_E \circ P_E = P_E$.

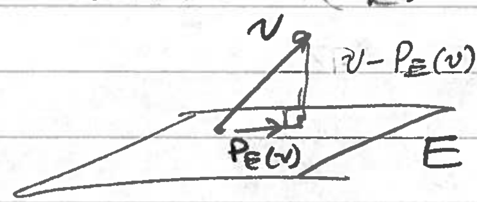
Also, $R(P_E) = \mathbb{C}w = E$, $N(P_E) = \{v \in V \mid (v,w) = 0\}$

$= \{v \in V \mid v \perp w\} = \{v \in V \mid v \perp \mathbb{C}w\}$.

$\Rightarrow N(P_E) \perp E$.

Lemma 28.2 Let V be a finite dimensional vector space, $E \subseteq V$ a subspace. Then $\exists$ an orthogonal projection $P_E : V \to V$ with $R(P_E) = E$

Picture



Note Since $v - P_E(v) \in N(P_E)$ $\quad v - P_E(v) \perp P_E(v)$

There are several ways to prove the lemma.

1) We can pick an orthonormal basis $\{v_1, \dots v_n\}$ of E

Then $P_E(v) = \sum_{j=1}^{n} (v, v_i) v_i$.

(see Thm 6.6 in Friedberg et al)

2) We define the orthogonal complement $E^\perp$ of E by

$$E^\perp = \{ v \in V \mid (v, w) = 0 \quad \forall w \in E \}.$$

Claim $E^\perp$ is a subspace of V and $E \cap E^\perp = \{0\}$.

Check $\forall \lambda_1, \lambda_2 \in \mathbb{C}, \quad v_1, v_2 \in E^\perp, \quad w \in E$

$$(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 (v_1, w) + \lambda_2 (v_2, w) = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0.$$

$$x \in E \cap E^\perp \Rightarrow (\underset{E^\perp}{x}, \underset{E}{x}) = 0 \Rightarrow x = 0.$$

Lemma 28.3 Let V be a finite dimensional inner product space, $E \subseteq V$ a subspace. Then $V = E \oplus E^\perp$

Proof We know by claim that $E \cap E^\perp = \{0\}$. So we need to show that $V = E + E^\perp$. Since $E \cap E^\perp = \{0\}$, enough to check that $\dim E + \dim E^\perp = V$.

Under the iso $\#: V \to \overline{V^*}$, $\#(v) = (\cdot, v)$

$$\#(E^\perp) = \{ l \in V^* \mid l(w) = 0 \quad \forall w \in E \} = E^0.$$

Anti-isomorphic vector spaces have the same dimensions

and $\dim E + \dim E^0 = \dim V$

$$\Rightarrow \dim E + \dim E^\perp = \dim V. \qquad \square$$

The direct sum decomposition $V = E \oplus E^\perp$ defines a projection $P_E : V \to V$ with $R(P_E) = E$, $N(P_E) = E^\perp$

Since $E \perp E^\perp$, P_E is orthogonal.

This proves lemma 28.2