Last time: real and complex vector spaces with inner products, Cauchy-Schwarz inequality:

$$|(x,y)| \leq \|x\| \, \|y\|$$

where $\|x\| = (x,x)^{1/2}$, $\|y\| = (y,y)^{1/2}$.

Aside A vector space V is complex or V is a vector space over $\mathbb{C}$ if the scalars are complex numbers. So the scalar multiplication is a map $\mathbb{C} \times V \to V$, $(\lambda, v) \mapsto \lambda v$

A vector space is real or a vector space over $\mathbb{R}$ if the scalar multiplication is a map

$$\mathbb{R} \times V \to V.$$

Note Any complex vector space V is also real since $\mathbb{R} \subseteq \mathbb{C}$

Note If V is complex and finite dimensional then

$$\dim_{\mathbb{C}} V = 2 \dim_{\mathbb{R}} V.$$

Reason if $\{v_1, \ldots v_n\}$ is a basis of a complex vector space V then $\{v_1, \sqrt{-1}\, v_1, v_2, \sqrt{-1}\, v_2, \ldots \; v_n, \sqrt{-1}\, v_n\}$ is a basis of V over $\mathbb{R}$. (Check that!)

Another complication Let V, W be two complex vector spaces.

A map $T: V \to W$ is $\mathbb{C}$-linear if $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $v_1, v_2 \in V$

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$$

A map $S: V \to W$ is antilinear if $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $v_1, v_2 \in V$

$$S(\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda_1}\, S(v_1) + \overline{\lambda_2}\, S(v_2)$$

Reinterpretation of antilinear: if W is a vector space $/\mathbb{C}$ we can change $\mathbb{C} \times W \to W$ into a new scalar multiplication:

$$\underset{\text{new}}{\lambda * w} := \underset{\text{old}}{\overline{\lambda} \cdot w}$$

This turns W into a different vector space over $\mathbb{C}$.
It is often denoted by $\overline{W}$.

<u>Note</u> $S: V \to W$ is <u>antilinear</u> $\Leftrightarrow$ $S: V \to \overline{W}$ is <u>linear</u>

<u>Example</u> Suppose $(\cdot,\cdot)$ is a Hermitian inner product on V
Then (1) $(x,y) = \overline{(y,x)}$ $\forall x,y \in V$
(2) $(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 (x_1,y) + \lambda_2 (x_2,y)$ $\forall \lambda_1,\lambda_2 \in \mathbb{C}$, $x_1,x_2,y \in V$
(3) $(x,x) \geq 0$ $\forall x \in V$ and
(4) $(x,x) = 0 \Leftrightarrow x = \vec{0}$.

Now ~~(2)~~ (2) says: $\forall$ ~~x~~ $y \in V$, $(\cdot, y) \in V^* = Hom(V,\mathbb{C})$.
(2)+(1) say: $\# : V \to V^*$, $\#(y) = (\cdot, y)$ is antilinear.
Hence $\# : V \to \overline{V^*}$ is linear.
(4) $\Rightarrow$ $\# : V \to \overline{V^*}$ is injective hence an iso if $\dim V < \infty$.

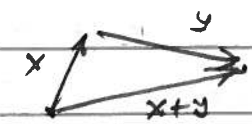
To conclude: if V is a finite dim v.space w. Hermitian inner product Then
$$\# : V \longrightarrow \overline{V^*} \qquad y \mapsto (-, y)$$
is an <u>iso</u> of complex vector spaces.

<u>Remark</u> i) If $\dim V = \infty$, $\#$ need not be an iso.
ii) Riesz representation theorem gives an analogue of $\#$ being an iso for Hilbert spaces.

An inner product $(\cdot,\cdot)$ on V defines a <u>norm</u> $\|\cdot\|$ on V by
$$\|x\| = (x,x)^{1/2}$$
Think: $\|x\|$ = length of x
Cauchy-Schwarz $\Rightarrow$
$$|(x,y)| \leq \|x\|\|y\| \qquad \forall x,y \in V.$$

<u>Lemma 27.1</u> (triangle inequality) Let V be a vector space with an inner product $(\cdot,\cdot)$. Then $\forall x, y \in V$

$$\|x+y\| \le \|x\| + \|y\|$$



Proof $\|x+y\|^2 = (x+y, x+y) = (x,x) + (x,y) + (y,x) + (y,y)$

$= \|x\|^2 + (x,y) + \overline{(x,y)} + \|y\|^2.$

<u>Recall</u> $\forall z \in \mathbb{C}$, (i) $z + \overline{z} = 2\mathrm{Re}(z)$

(ii) $\mathrm{Re}(z) \le |\mathrm{Re}(z)| \le \left(\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2\right)^{1/2} = |z|$

Hence

$$\|x+y\|^2 \le \|x\|^2 + 2|(x,y)| + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$\uparrow$ Cauchy-Schwarz

$$\Rightarrow \quad \|x+y\| \le \left((\|x\|+\|y\|)^2\right)^{1/2} = \|x\| + \|y\|. \qquad \square$$

<u>Def</u> Let V be a vector space with an inner product. Two vectors $x, y \in V$ are <u>orthogonal</u> $\Leftrightarrow$ $(x,y) = 0$.

We write $x \perp y$ if x & y are orthogonal.

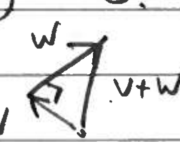
<u>Ex</u> $\begin{pmatrix}1\\ i\end{pmatrix}, \begin{pmatrix}1\\ -i\end{pmatrix} \in \mathbb{C}^2$ are orthogonal:

$$\left(\begin{pmatrix}1\\ i\end{pmatrix}, \begin{pmatrix}1\\ -i\end{pmatrix}\right) = 1\cdot\overline{1} + i\cdot\overline{(-i)} = 1^2 + i^2 = 1 - 1 = 0.$$

<u>Ex</u> $e^{it}, e^{2it} \in C^0([0,2\pi], \mathbb{C})$ are orthogonal:

$$(e^{2it}, e^{it}) = \int_0^{2\pi} e^{2it}\cdot\overline{e^{it}}\,dt = \int_0^{2\pi} e^{2it}\cdot e^{-it}\,dt$$

$$= \int_0^{2\pi} e^{it}\,dt = \frac{1}{i}e^{it}\Big|_0^{2\pi} = \frac{1}{i}\left(e^{2\pi i} - e^0\right) = 0.$$



<u>Remark</u> If $(v,w) = 0$ Then $\|x+w\|^2 = \|v\|^2 + \|w\|^2$.

<u>Proof</u> $\|v+w\|^2 = (v+w, v+w) = (v,v) + (v,w) + (w,v) + (w,w)$

$= \|v\|^2 + \underbrace{(v,w)}_{0} + \underbrace{\overline{(v,w)}}_{\overline{0}} + \|w\|^2 = \|v\|^2 + \|w\|^2$

"Pythagoras theorem by definition"

Def A collection of vectors $\{v_1, \ldots, v_n\}$ in a vector space V w. an inner product is **orthogonal** if $(v_i, v_j) = 0$ for $i \neq j$
and $v_i \neq 0\ \forall i$ (so $(v_i, v_i) = \|v_i\|^2 \neq 0$)
A collection of vectors $\{v_1 \ldots v_n\}$ is **orthonormal** if
$(v_i, v_j) = \delta_{ij}\ \forall i, j$ (so $\|v_i\| = 1$ for all i)

Note $\{v_1 \ldots v_n\}$ orthogonal $\Rightarrow \left\{ \frac{v_1}{\|v_1\|}, \ldots \frac{v_n}{\|v_n\|} \right\}$ orthonormal.
Reason: $\forall j$

$$\left(\frac{v_j}{\|v_j\|}, \frac{v_j}{\|v_j\|} \right) = \frac{1}{\|v_j\|} \cdot \frac{1}{\|v_j\|} \cdot (v_j, v_j) = \frac{\|v_j\|^2}{\|v_j\|^2} = 1.$$

Lemma 27.2 An orthogonal collection $\{v_1, \ldots, v_n\}$ of vectors is linearly independent.
Proof Suppose $\exists a_1 \ldots a_n \in \mathbb{C}$ st $\sum_{j=1}^{n} a_j v_j = 0$. Then $\forall k$

$$0 = (0, v_k) = \left(\sum_{j=1}^{n} a_j v_j, v_k \right) = \sum_{j=1}^{n} a_j (v_j, v_k) = a_k (v_k, v_k).$$

since $(v_j, v_k) = 0$ for $j \neq k$. Since $(v_k, v_k) \neq 0$, $a_k = 0$. $\square$

Lemma 27.3 (cf Treil, p186 Lemma 3.5) Suppose $\{v_1 \ldots v_n\} \subseteq V$ is orthogonal. Then $\forall a_1 \ldots a_n \in \mathbb{C}$

$$\left\| \sum_{j=1}^{n} a_j v_j \right\|^2 = \sum_{j=1}^{n} |a_j|^2 \|v_j\|^2$$

Proof

$$\left(\sum_j a_j v_j, \sum_k a_k v_k \right) = \sum_{j,k} a_j \overline{a_k} (v_j, v_k) = \sum_{j=1}^{n} a_j \overline{a_j} (v_j, v_j)$$

since $(v_j, v_k) = 0$ for $j \neq k$.

$\therefore$

$$\left\| \sum a_j v_j \right\|^2 = \sum |a_j|^2 \|v_j\|^2.$$