# Inner products (mostly following Treil)

**Definition** A (Hermitian) inner product on a complex vector space $V$ is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ so that

(1) $(y, x) = \overline{(x, y)}$ for all $x, y \in V$

(2) $\forall\, \lambda, \mu \in \mathbb{C},\ x, y, z \in V \qquad (\lambda x + \mu y, z) = \lambda (x, z) + \mu (y, z)$

i.e. $(\cdot,\cdot)$ is $\mathbb{C}$-linear in the left ($1^{st}$) slot)

(3) $(x, x) \geq 0 \quad \forall x \in V$ (note: since $(x,x) = \overline{(x,x)}$, $(x,x) \in \mathbb{R}$)

(4) $(x, x) = 0 \iff x = \vec{0}$. (nondegeneracy)

**Basic example** $V = \mathbb{C}^n \qquad (x, y) = \sum_{j=1}^{n} x_j \overline{y_j}$

**Remark** (1) + (2) $\Rightarrow$

$$(z, \lambda x + \mu y) = \overline{(\lambda x + \mu y, z)} = \overline{\lambda (x, z)} + \overline{\mu (y, z)}$$
$$= \bar{\lambda}\, \overline{(x, z)} + \bar{\mu}\, \overline{(y, z)} = \bar{\lambda}\, (z, x) + \bar{\mu}\, (z, y)$$

we say $(\cdot,\cdot)$ is anti-linear in the second (right) slot.

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**WARNING** In physics the convention is:

$(\cdot,\cdot)$ is linear in the $2^{nd}$ (right) slot and anti-linear in the first. In particular, $\forall\, x \in V$

$(x, \cdot) : V \to \mathbb{C}$ is $\mathbb{C}$-linear

Dirac notation: $\langle x| := (x, \cdot) \in \mathrm{Hom}^*(V, \mathbb{C})$

In math $(\cdot, x) \in \mathrm{Hom}(V, \mathbb{C}) \quad \forall\, x \in V$.

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**Example 2** $V = \{ f : [0, 2\pi] \to \mathbb{C} \mid f \text{ continuous} \}$

$\forall f, g \in V, \quad (f, g) := \int_0^{2\pi} f(t)\, \overline{g(t)}\, dt$

**Note** $(f, f) = \int_0^{2\pi} f(t) \overline{f(t)}\, dt = \int_0^{2\pi} |f(t)|^2 dt \geq 0$

**and** $(f, f) = 0 \iff f(t) = 0 \quad \forall t$. (requires some analysis to prove)

<u>Note</u> if $f(t) = e^{ikt}$, $g(t) = e^{imt}$ $(k, m \in \mathbb{Z})$

$$(f,g) = \int_0^{2\pi} e^{kit} \cdot \overline{e^{mit}}\, dt = \int_0^{2\pi} e^{ikt} \cdot e^{-imt}\, dt$$

$= \int_0^{2\pi} e^{i(k-m)t}\, dt$. If $k = m$, we get $\int_0^{2\pi} e^0\, dt = 2\pi \cdot e^0 = 2\pi$

If $k \neq m$, we get $\frac{1}{i(k-m)} e^{(k-m)it} \Big|_0^{2\pi} = \frac{1}{i(k-m)} \left( e^{2\pi(k-m)i} - e^0 \right) = 0$

(remember: $e^{ix} = \cos x + i \sin x$ $\forall x \in \mathbb{R}$, so $e^{2\pi i \ell} = 1$ $\forall \ell \in \mathbb{Z}$)

<u>Lemma 26.1</u> (Treil, ~p.118) Let $V$ be a vector space with an inner product $(\cdot,\cdot)$. Then $x = \vec{0} \iff (x,y) = 0$ $\forall y \in V$.

<u>Proof</u> $(\Rightarrow)$ $(\vec{0}, y) = (0 \cdot \vec{0}, y) = 0 \cdot (\vec{0}, y) = 0$.

$(\Leftarrow)$ if $(x,y) = 0$ $\forall y \in V$, then $(x,x) = 0 \Rightarrow x = \vec{0}$.

<u>Interpretation</u> $\forall x \in V$, $(\cdot, x) : V \to \mathbb{C}$, $y \mapsto (y, x)$ is $\mathbb{C}$-linear.

We therefore have a map

$$\# : V \to V^*, \quad \#(x) = (\cdot, x).$$

The map $\#$ is $\mathbb{C}$-antilinear:

$$\left( \#(\lambda_1 x_1 + \lambda_2 x_2) \right)(y) = (y, \lambda_1 x_1 + \lambda_2 x_2)$$
$$= \overline{\lambda_1}(y, x_1) + \overline{\lambda_2}(y, x_2)$$
$$= \left( \overline{\lambda_1}(\# x_1) + \overline{\lambda_2}(\# x_2) \right)(y).$$

In particular, $\#$ is $\mathbb{R}$-linear. (any vector space over $\mathbb{C}$ is a vector space over $\mathbb{R}$ since $\mathbb{R} \subseteq \mathbb{C}$)

$N(\#) = \{ x \in V \mid (y,x) = 0 \ \forall y \in V \}$.

By 26.1, $N(\#) = \{0\}$, $\Rightarrow$ $\#$ is injective.

In particular: if $(z,x) = (z,y)$ $\forall z \in V$ then $x = y$.

<u>Remark</u> If $\dim V < \infty$, then $\#$ is an isomorphism of <u>real</u> vector spaces. $\#$ is not a $\mathbb{C}$-linear map since

$$\#(\lambda x) = \overline{\lambda}\, \#(x) \quad \forall x \in V.$$

Corollary 26.2 Suppose $A, B: W \to V$ are two $\mathbb{C}$-linear maps $(\cdot,\cdot)$ an inner product on $V$ and

$$(Ax, y) = (Bx, y) \quad \forall x \in W, \forall y \in V.$$

Then $A = B$.

Proof $(Ax, y) = (Bx, y)$ $\forall y, \forall x \Rightarrow Ax = Bx$ $\forall x \Rightarrow A = B$ ☐

Definition Let $V$ be a complex vector space with an inner product $(\cdot,\cdot)$. The norm of $x \in V$ is

$$\|x\| := (x, x)^{1/2}$$

Ex $V = \{ f: [0, 2\pi] \to \mathbb{C} \mid f \text{ continuous} \}$

$$\|f\| = \left(\int_0^{2\pi} f(t)\bar{f}(t)\,dt\right)^{1/2} = \int_0^{2\pi} |f(t)|^2 dt$$

is the norm of $f$ w.r.t. $(f, g) = \int_0^{2\pi} f(t)\bar{g}(t)\,dt$.

Theorem 26.3 (Cauchy-Schwarz inequality) Let $V$ be a vector space with an inner product $(\cdot,\cdot)$. Then

$$(\star) \quad |(x, y)| \leq \|x\| \, \|y\| \quad \forall x, y \in V.$$

Proof If $y = \vec{0}$, $(x, y) = 0$ for any $x$ and $\|y\| = (0,0)^{1/2} = 0$. So $(\star)$ holds.

Suppose $y \neq 0$. Then $0 \neq (y, y) = \|y\|^2$ (In fact $\|y\|^2 > 0$.)

For any $t \in \mathbb{C}$

$$0 \leq \|x - ty\|^2 = (x - ty, x - ty) = (x, x - ty) - t(y, x - ty)$$

$$= (x, x) - \bar{t}(x, y) - t(y, x) + t\bar{t}(y, y)$$

Now set $t = \frac{(x, y)}{\|y\|^2}$. We get

$$0 \leq \|x\|^2 - \frac{\overline{(x,y)}}{\|y\|^2}(x, y) - \frac{(x,y)}{\|y\|^2}(y, x) + \left|\frac{(x,y)}{\|y\|^2}\right|^2 \|y\|^2$$

$$= \|x\|^2 - \cancel{\frac{|(x,y)|^2}{\|y\|^2}} - \frac{(x,y)\overline{(x,y)}}{\|y\|^2} + \cancel{\frac{|(x,y)|^2}{\|y\|^4}\|y\|^2}$$

$$= \|x\|^2 - \frac{|(x, y)|^2}{\|y\|^2}. \Rightarrow \frac{|(x,y)|^2}{\|y\|^2} \leq \|x\|^2.$$

Since $\|y\|^2 > 0$, we get $|(x,y)| \le \|x\|^2 \cdot \|y\|^2$.

<u>Triangle inequality</u> $\forall\, x, y \in V$

$$\|x+y\| \le \|x\| + \|y\|.$$

<u>Proof</u>

$$\|x+y\|^2 = (x+y, x+y) = (x,x) + (y,x) + (x,y) + (y,y)$$
$$= \|x\|^2 + \overline{(x,y)} + (x,y) + \|y\|^2$$

For any $z \in \mathbb{C}$

$$z + \bar{z} = 2\,\mathrm{Re}(z) \le 2\,|\mathrm{Re}(z)| \le 2\,|z|$$

$$\Rightarrow \quad \|x+y\|^2 \le \|x\|^2 + 2\,|(x,y)| + \|y\|^2$$
$$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{by Cauchy-Schwarz}$$
$$= (\|x\| + \|y\|)^2.$$
$$\therefore \quad \|x+y\| \le \|x\| + \|y\|. \qquad \square$$

Definition Let $V$ be a vector space with an inner product $(\cdot,\cdot)$

Two vectors $x, y \in V$ are <u>othogonal</u> if $(x,y) = 0$.

<u>Notation</u> $x \perp y \iff (x,y) = 0$

<u>Ex</u> $\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{C}^2$ are orthogonal.

<u>Ex</u> $e^{it}$ and $e^{2it}$ are orthogonal in $V = \{f : [0,2\pi] \to \mathbb{C} \mid f$ continuous$\}$

<u>Def</u> A basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of $V$ is <u>orthogonal</u> if

$$(b_i, b_j) = 0 \quad \text{for } i \neq j.$$

It's <u>orthonormal</u> if

$$(b_i, b_j) = \delta_{ij} \quad \forall\, i, j.$$