HW #10 & review sheet 2 are posted.

03/27/2019
We
25.1

Last time

Defined the characteristic polynomial $p_T(\lambda)$ of a linear map $T: V \to V$;

$$p_T(\lambda) := \det(\lambda\, id - T) = \det(\lambda I - [T]_{BB})$$

where B is a basis of V.

Def A polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is monic if $a_n = 1$

Note $p(x) \in \mathbb{C}[x]$ is monic, $\lambda_1 \dots \lambda_n$ roots of $p(x)$ (with multiplicities) $\Rightarrow$

$$p(x) = (x-\lambda_1)(x-\lambda_2)\cdot\ldots\cdot(x-\lambda_n)$$

(in general if $h(x) \in \mathbb{C}[x]$ and $\deg h = n$, then

$$h(x) = c\cdot(\lambda-\lambda_1)\ldots(\lambda-\lambda_n)$$
$$= c\cdot\lambda^n + \text{lower order terms}, \quad c \neq 0 \text{)}$$

$$\Rightarrow \quad p(x) = x^n - \left(\sum \lambda_j\right) x^{n-1} \quad + \quad (-1)^n \lambda_1 \cdot\ldots\cdot \lambda_n.$$

Lemma 25.1 Let A be an $n\times n$ complex matrix, $\lambda_1 \dots \lambda_n$ corresponding eigenvalues (roots of $p_A(x) = \det(xI - A)$)

Then (1) $\prod_{j=1}^{n} \lambda_j = \det A$

(2) $\sum_{j=1}^{n} \lambda_j = \operatorname{tr} A$.

Proof (1) $p_A(x) = \det(xI - A) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n)$

$$= x^n + \left(-\sum_{j=1}^{n} \lambda_j\right) x^{n-1} + \cdots + (-1)^n \lambda_1 \ldots \lambda_n$$

$$\Rightarrow \quad p_A(0) = (-1)^n \lambda_1 \ldots \lambda_n.$$

On the other hand $p_A(0) = \det(0\cdot I - A) = \det(-A) = (-1)^n \det A$

$$\therefore \quad \det A = \lambda_1 \cdot\ldots\cdot \lambda_n.$$

(2) The key is to observe that the coefficient of x^{n-1} in

$$p_A(x) = \det \underbrace{\begin{pmatrix} x-a_{11} & -a_{12} & \cdots \\ -a_{21} & x-a_{22} & \\ \vdots & & \end{pmatrix}}_{b_{ij}}$$

is $-a_{11} - a_{22} - \cdots - a_{nn}$

$$\det(b_{ij}) = \sum_{\sigma \in S_n} (\operatorname{sign} \sigma)\, b_{\sigma(1)1} \cdots b_{\sigma(n)n}$$

This works because to get x^{n-1} we need to pick $n-1$ diagonal terms. Say we picked $b_{11}, b_{22} \ldots b_{n-1\,n-1}$. This forces the last term in the product to be b_{nn}

$b_{ii} = (x - a_{ii})$. $\Rightarrow$ we get $x \cdot x \cdot \ldots x \cdot (-a_{nn})$

If we pick $b_{11}, b_{22}, \ldots b_{n-2,n-2}, b_{n,n}$, we get

$$\underbrace{x \cdots x}_{n-2} \cdot (-a_{n-1,n-1}) \cdot x \qquad \text{etc.}$$

<u>Definition</u> A linear map $T: V \to V$ is <u>diagonalizable</u> if there is a basis $\mathcal{B}$ of V s.t. $[T]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

for some scalars $\lambda_1 \ldots \lambda_n$.

A matrix $A \in M_{n,n}$ is <u>diagonalizable</u> if there is an invertible matrix S so that

$$SAS^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

<u>Exerise</u> A matrix A is diagonalizable $\Leftrightarrow$ the corresponding linear map A is diagonalizable

<u>Hint</u> • $A = [A]_{\mathcal{S}\mathcal{S}}$, $\mathcal{S}$ standard basis.

• $[A]_{\mathcal{B}\mathcal{B}} = [id]_{\mathcal{B}\mathcal{S}} [A]_{\mathcal{S}\mathcal{S}} [id]_{\mathcal{S}\mathcal{B}}$

<u>Remark</u> T is diagonalizable (in a basis $\mathcal{B} = \{b_1 \ldots b_n\}$)

$$T b_i = \lambda b_i \quad \forall i$$

$\Leftrightarrow$ $\mathcal{B}$ is a basis of eigenvalues of T.

<u>Theorem 25.2</u> A sufficient condition for a linear map $T: V \to V$ to be diagonalizable is that T has $n = \dim$ <u>distinct</u> eigenvalues $\lambda_1 \ldots \lambda_n$.

25.2 is a consequence of

Lemma 25.3 Suppose $T: V \to V$ is linear, $\lambda_1, \dots \lambda_k$ distinct eigenvalues. Then the set of corresponding eigenvectors $\{v_1, \dots v_k\}$ is linearly independent.

Proof Suppose not. Then a subset of size $r < k$ of $\{v_1, \dots v_k\}$ is linearly independent. By renumbering v_i's we may assume: $\{v_1, \dots v_r\}$ is linearly independent. Then

$$v_{r+1} = a_1 v_1 + \dots + a_r v_r \quad \text{for some } a_1, \dots a_r \ (\underline{\underline{\text{not all zero}}})$$

Apply T to both sides. We get

$$\lambda_{r+1} v_{r+1} = T(v_{r+1}) = T\Big(\sum_{i=1}^{r} a_i v_i\Big) = \sum_{i=1}^{r} a_i \lambda_i v_i$$

$$\Rightarrow 0 = \lambda_{r+1} v_{r+1} - \lambda_{r+1} v_{r+1} = \lambda_{r+1}\Big(\sum_{i=1}^{r} a_i v_i\Big) - \sum_{i=1}^{r} (a_i \lambda_i v_i)$$
$$= \sum_{i=1}^{r} (\lambda_{r+1} - \lambda_i) a_i v_i.$$

Since $\{v_1, \dots v_r\}$ is lin. independent

$$\begin{cases} (\lambda_{r+1} - \lambda_1)\, a_1 = 0 \\ (\lambda_{r+1} - \lambda_2)\, a_2 = 0 \\ \vdots \\ (\lambda_{r+1} - \lambda_r) \cdot a_r = 0 \end{cases}$$

$\lambda_{r+1} - \lambda_i \neq 0$ for all $i = 1, \dots r$. $\Rightarrow a_1 = a_2 \dots = a_r = 0$

Contradiction. $\blacksquare$

Remark The only eigenvalue of $id_V : V \to V$ is 1.

And in any basis $\mathcal{B}$, $[id_V]_{\mathcal{B}\mathcal{B}} = I$

So having distinct eigenvalues is not necessarily for diagonalizability

Why diagonalize?

Ex 1 Suppose $A = B\Lambda B^{-1}$ where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$

Then

$$A^n = \underbrace{(B\Lambda B^{-1})(B\Lambda B^{-1})\cdots(B\Lambda B^{-1})}_{n} = B\Lambda^n B^{-1}$$

$$= B\begin{pmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{pmatrix} B^{-1}$$

So powers are easy to compute.

Ex 2 Recall $\forall x \in \mathbb{R}$ (or $\forall x \in \mathbb{C}$)

$$e^x = 1 + \tfrac{1}{1!}x + \tfrac{1}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

For any $A \in M_{n,n}$ we define $e^A = I + A + \frac{1}{2!}A^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

$\left(= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n!} A^n \right)$

One can show: e^A exists for all $A \in M_{n,n}$.

Also $\frac{d}{dt}(e^{tA}) = Ae^{tA}$

So $\forall x \in \mathbb{R}^n$, $\forall A \in M_{n,n}(\mathbb{R})$ $f(t) := e^{tA}x$ solves

$$\begin{cases} \frac{d}{dt} f = Af \\ f(0) = x \end{cases}$$

If A is diagonalizable, e^{tA} is "easy" to compute!

if $A = B\Lambda B^{-1}$, where $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$ then

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} (B\Lambda B^{-1})^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} B\Lambda^n B^{-1}$$

$$= B\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda^n\right) B^{-1} = B\begin{pmatrix} \sum \frac{t^n}{n!}\lambda_1^n & & \\ & \ddots & \\ & & \sum \frac{t^n}{n!}\lambda_k^n \end{pmatrix} B^{-1}$$

$$= B\begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix} B^{-1}.$$