Last time • Defined trace of a square matrix and of a linear map.
• Defined eigenvalues and eigenvectors.

Recall:

Suppose V is a vector space over $\mathbb{C}$ (the scalars are complex numbers and $T: V \to V$ is ($\mathbb{C}$-) linear (ie

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2) \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}, v_1, v_2 \in V$$

$\lambda \in \mathbb{C}$ is an eigenvalue of T if $\exists v \in V$, $v \neq 0$ s.t.

$$T(v) = \lambda v$$

We say: "v is an eigenvector of T with eigenvalue λ"

Note: $\forall a \in \mathbb{C}$

$$T(av) = aT(v) = a\lambda v = \lambda(av)$$

So if v is an eigenvector w. eigenvalue λ so is av, $\forall a \in \mathbb{C}, a \neq 0$

Definition The set of eigenvalues of a map $T: V \to V$ is called the spectrum of T. One writes: $\operatorname{spec}(T)$ or $\sigma(T)$.

Ex $V = C^\infty(\mathbb{R})$ = infinitely differentiable real-valued functions of one variable; it's a vector space over $\mathbb{R}$.

$T = \frac{d}{dx} : V \to V$

Claim $\operatorname{spec}(\frac{d}{dx}) = \mathbb{R}$

Reason Fix $\lambda \in \mathbb{R}$. Suppose $\frac{df}{dx} = \lambda f$

Then $$\frac{1}{f}\frac{df}{dx} = \lambda$$

$$\Rightarrow \frac{d}{dx}(\ln f) = \lambda$$

$$\Rightarrow \int_0^x \frac{d}{ds}(\ln f(s))\,ds = \int_0^x \lambda\, ds$$

$$\Rightarrow \ln f(x) - \ln f(0) = \lambda x$$

$$\Rightarrow f(x) = f(0)e^{\lambda x}$$ which is an element of $C^\infty(\mathbb{R})$

and $\frac{d}{dx} f(0)e^{\lambda x} = \lambda(f(0)e^{\lambda x})$.

We've seen last time $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$
has no real eigenvalues. (unless $\theta = 0, \pi$)
However if we think of A as a map $\mathbb{C}^2 \to \mathbb{C}^2$
then A has two complex eigenvalues: $\lambda = e^{i\theta}$, $\lambda = e^{-i\theta}$.

Lemma 24.1 Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$), $T: V \to V$ a linear map. Then
λ is an eigenvalue of T
$\iff \dim(N(\lambda\, id_V - T)) > 0$ $\quad$ ($id_V : V \to V$ $\quad id_V(u) = u \; \forall u \in V$)
Moreover, if $\dim V < \infty$
$\dim(N(\lambda\, id_V - T)) > 0 \iff \det(\lambda\, Id_V - T) = 0$.

Proof λ is an eigenvalue of $T \iff$
$\exists\, 0 \neq v \in V$ s.t. $Tv = \lambda v$
$\iff Tv = (\lambda \cdot id_V) v$
$\iff (\lambda\, id_V - T)(v) = 0$
$\iff v \in N(\lambda\, id_V - T)$
$\iff \dim(N(\lambda\, id_V - T)) > 0$
Now, if V is finite dimensional and $S: V \to V$ is linear
$\dim N(S) > 0 \iff \det S = 0$
Therefore, λ is an eigenvalue of $T \iff \det(\lambda\, id_V - T) = 0$.
$\square$

Aside
Recall $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R}$
(or $\det : \mathbb{C}^n \times \dots \times \mathbb{C}^n \to \mathbb{C}$) is n-linear. So
$\forall\, A = (a_1 | a_2 | \dots | a_n) \in M_{n \times n}$, $\quad \lambda A = (\lambda a_1 | \lambda a_2 | \dots | \lambda a_n)$
$\Rightarrow \boxed{\det(\lambda A) = \lambda^n \cdot \det A.}$

In particular, $\forall A \in M_{n,n}$ $\det(\lambda I - A) = \det((-1)(A - \lambda I))$
$= (-1)^n \det(A - \lambda I)$.

Remark Let V be an n-dim vector space over $\mathbb{C}$ and $T: V \to V$ a linear map. Then the function

$$p_T(\lambda) = \det(\lambda\, id_V - T) = \det([\lambda\, id_V - T]_{BB})$$
$$= \det(\lambda I - [T]_{BB}) \qquad B = \text{basis of } V$$

is a polynomial in λ of degree n.

The reason is that for any $n \times n$ matrix $A \in M_{n,n}(\mathbb{C})$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & & \vdots \\ \vdots & & \ddots & \\ & & & \lambda - a_{nn} \end{pmatrix}$$

$= (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) +$ other products with sign that may or may not involve $(\lambda - a_{ii})$ terms.

$= \lambda^n +$ lower order terms.

(Recall $\det(c_{ij}) = \sum_{\sigma \in S_n} (\text{sign}\,\sigma)\, c_{\sigma(1)1} \cdots c_{\sigma(n)n}$)

Definition The characteristic polynomial $p_T(\lambda)$ of a linear map $T: V \to V$ ($\dim_{\mathbb{C}} V = n$) is

$$p_T(\lambda) := \det(\lambda\, id_V - T)$$

Remark The textbook defines the characteristic polynomial of $T: V \to V$ to be $\det(T - \lambda\, id_V)$,
which is $(-1)^{\dim V} \det(\lambda\, id_V - T) = (-1)^{\dim V} p_T(\lambda)$.

We need to "recall" a few facts about polynomials.

- Every (nonconstant) complex polynomial $p(x) \in \mathbb{C}[x]$ has a root: $\exists\, \lambda_0 \in \mathbb{C}$ s.t. $p(\lambda_0) = 0$.

This is false for real polynomials: $p(x) = x^2+1$ has no real roots.

- if $\lambda_0 \in \mathbb{C}$ is a root of $p(x) \in \mathbb{C}[x]$ then

$$p(x) = (x-\lambda_0)\, q(x)$$

for some polynomial $q(x)$ with $\deg q = \deg p - 1$.

Induction on $\deg p \Rightarrow$

$$p(x) = a(x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdots (x-\lambda_k)^{n_k}$$

for some $k \in \mathbb{N}$, $a \neq 0$, $\lambda_1 \dots \lambda_k \in \mathbb{C}$, $n_1 \dots n_k \in \mathbb{N}$ with

$$n_1 + n_2 + \cdots + n_k = \deg p.$$

n_i = multiplicity of the root λ_i

Conclusion Suppose V is a vector space over $\mathbb{C}$, $T: V \to V$ linear, $n = \dim V$. Then $p_T(\lambda) = \det(\lambda \mathrm{Id}_V - T)$ has n roots (counted with multiplicities). Hence T has n eigenvalues (counted with multiplicities.

Ex $T = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ $p_T(\lambda) = \det(\lambda I - T) = \det\begin{pmatrix} \lambda-3 & -1 \\ 0 & \lambda-3 \end{pmatrix}$
$= (\lambda-3)^2$.

$\lambda = 3$ is the eigenvalue of T (with algebraic multiplicity 2)

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of T $\Leftrightarrow$

$\wedge /$

$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Leftrightarrow$ $-v_2 = 0$ (and $v_1 \neq 0$) $\Rightarrow$ $v = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $c \neq 0$

check

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note $\dim(N(3I - T)) = 1 \neq 2!$

Next time If $A \in M_{n,n}(\mathbb{C})$ has n distinct eigenvalues then A has a basis of eigenvectors.