Last time Given an $n\times n$ matrix $A = (a_{rs})$

$A_{jk} = (n-1)\times(n-1)$ matrix obtained by deleting from A the row and column containing a_{ij}.

The matrix of cofactors C of A has entries

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

22.3 (Thm 5.2 in Treil)

$$AC^T = (\det A)\cdot I$$

We also proved.

22.1 An $n\times n$ matrix A is invertible $\iff \det A \neq 0$.

Def Two $n\times n$ matrices A & B are similar iff there is an invertible matrix S so that

$$A = SBS^{-1}.$$

"Example" V = n-dim vector space, $T: V \to V$ linear map

$\mathcal{A}$, $\mathcal{B}$ two bases. Then $[T]_{\mathcal{A}\mathcal{A}}$ and $[T]_{\mathcal{B}\mathcal{B}}$ are similar:

$$[T]_{\mathcal{A}\mathcal{A}} = [id]_{\mathcal{A}\mathcal{B}}\,[T]_{\mathcal{B}\mathcal{B}}\,[id]_{\mathcal{B}\mathcal{A}}$$

Lemma 22.4 If $A = SBS^{-1}$ then $\det A = \det B$.

Proof

$$\det A = \det SBS^{-1} = \det S \det B \det S^{-1}$$
$$= \det S \det S^{-1} \det B = \det(SS^{-1})\det B$$
$$= \det I \det B = 1\cdot \det B.$$

$\square$

Corollary 23.1 Given a linear map $T: V \to V$ and a basis $\mathcal{B}$ of V, $\det [T]_{\mathcal{B}\mathcal{B}}$ does not depends on the choice of the basis $\mathcal{B}$: $\forall$ basis $\mathcal{A}$

$$\det [T]_{\mathcal{B}\mathcal{B}} = \det [T]_{\mathcal{A}\mathcal{A}}$$

Aside a "basis free" definition of $\det T$, $T: V \to V$
$\dim V = n < \infty$:

(i) Since $\dim V = n$, $V \cong \mathbb{R}^n$ ("$\cong$" = "isomorphic to")
$\Rightarrow \mathrm{Alt}^n(V) \cong \mathrm{Alt}^n(\mathbb{R}^n)$
$\dim \mathrm{Alt}^n(V) = \dim \mathrm{Alt}^n(\mathbb{R}^n) = 1$.

(ii) $T: V \to V$ defines $T^*: \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$;
T^* is linear $\dim \mathrm{Alt}^n(V) = 1$
$\exists$ a constant $c = c(T)$ s.t. $\forall \alpha \in \mathrm{Alt}^n(V)$
$$T^*\alpha = c \cdot \alpha$$
One can <u>define</u> $\det T = c$.

<u>Sanity check:</u>
$V = \mathbb{R}^n$, $T: \mathbb{R}^n \to \mathbb{R}^n$ linear map corr. to the matrix A: $A = [T]_{\mathcal{A}\mathcal{A}}$ $\mathcal{A}$ = standard basis
$$(T^*\det) = \det A \cdot \det$$
so the two definitions agree in the case of $V = \mathbb{R}^n$.

Trace Let $A = (a_{ij})$ be an $n \times n$ matrix. We define its trace to be
$$\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}.$$

<u>Ex</u> $$\mathrm{tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9.$$

<u>Proposition 23.2</u> (i) $\forall$ $n \times n$ matrices A and B
$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$
(ii) if $C = SDS^{-1}$ then $\mathrm{tr}\, C = \mathrm{tr}\, D$.

<u>Proof</u> (i)
$$\mathrm{tr}(AB) = \sum_{i=1}^n (AB)_{ii} = \sum_{i=1}^n \left(\sum_{j=1}^n A_{ij} B_{ji}\right) = \sum_{j=1}^n \sum_{i=1}^n B_{ji} A_{ij}$$
$$= \sum_{j=1}^n (BA)_{jj} = \mathrm{tr}(BA).$$

(ii) $\operatorname{tr}((SD)S^{-1}) = \operatorname{tr}(S^{-1}(SD)) = \operatorname{tr}(ID) = \operatorname{tr} D.$

__Corollary 23.3__ Let V be an n-dim V space, $T: V \to V$ a linear map and B a basis of V. Then

$$\operatorname{tr} T := \operatorname{tr} [T]_{BB}$$

does not depend on the choice of B

__Proof__ If A is any other basis

$$[T]_{AA} = [id]_{AB} [T]_{BB} ([id]_{AB})^{-1}.$$

$\square$

Eigenvectors and eigenvalues

__Def__ Let $T: V \to V$ be a linear map. An eigenvector of T with eigenvalue λ is a __nonzero__ vector $v \in V$ so that

$$T(v) = \lambda v.$$

__Ex__ $V = \mathbb{R}^2$ $\quad T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$T e_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2e_1.$$

$$T e_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3e_2$$

e_1 is an eigenvector of T with eigenvalue 2

e_2 —— // —— 3.

__Note__ $T\vec{0} = \vec{0} = \lambda \cdot \vec{0} \quad \forall \lambda \in \mathbb{R}$

This is why we require eigenvectors to be __nonzero__.

Ex $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ what are the eigenvectors and the corresponding eigenvalues?

Solution $Av = \lambda v \iff$

$Av = \lambda I v \iff (A - \lambda I)\, v = 0$

$\iff N(A - \lambda I) \neq \{0\}$

$\iff \det(A - \lambda I) = 0$

$\det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$

$\Rightarrow \lambda = \pm 1$

If $\lambda = 1$ $A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $N\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = ?$

$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow -x_1 + x_2 = 0 \rightsquigarrow x_1 = x_2$

$\Rightarrow v = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in N\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$\rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalue 1.

(any nonzero multiple of v_1 is <u>also</u> an eigenvector)

Similarly one computes that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenvalue -1.

Ex $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$0 = \det(A - \lambda I) = \det\begin{pmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{pmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta$

$= (\lambda - \cos\theta)^2 - (\sqrt{-1}\,\sin\theta)^2$

$= (\lambda - \cos\theta - \sqrt{-1}\,\sin\theta)(\lambda - \cos\theta + \sqrt{-1}\,\sin\theta)$

"<u>Recall</u>" $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \lambda = e^{i\theta}, e^{-i\theta}$ are eigenvalues of A. So if $\theta \neq 0, \pi$ no <u>real</u> eigenvalues!

Computation $\rightsquigarrow$

$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ are the corresponding eigenvectors.