Last time

Proved $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n = M_{n,n}(\mathbb{R}) \to \mathbb{R}$

$\det((a_{ij})) = \sum_{\sigma \in S_n} \operatorname{sign}\sigma \prod_{i=1}^{n} a_{\sigma(i) i}$ is alternating

[The proof has a mistake which was spotted by Abhishek Desai:
if (b_{ij}) is obtained from (a_{ij}) by switching k^{th} and ℓ^{th} columns
then $b_{ij} = a_{i\tau(j)}$ where $\tau = (k\ \ell)$ the transposition that switches k and ℓ.

So $\det(b_{ij}) = \sum_{\sigma \in S_n} (\operatorname{sign}\sigma) \prod_{i=1}^{n} b_{\sigma(i) i} = \sum_{\sigma \in S_n} \operatorname{sign}\sigma \prod_{i=1}^{n} a_{\sigma(i)\tau(i)}$

(change of variables: $j = \tau(i)$, $i = \tau(j)$)

$= \sum_{\sigma \in S_n} \operatorname{sign}\sigma \prod_{j=1}^{n} a_{\sigma(\tau(j)) j} = \left(\begin{matrix} \mu := \sigma\tau \\ \operatorname{sign}\mu = (-1)\operatorname{sign}\sigma \end{matrix} \right)$

$= \sum_{\mu \in S_n} (-1) \operatorname{sign}\mu \prod_{j=1}^{n} a_{\mu(j) j} = -\det((a_{ij})).$ ◻

We also proved: 21.1 $\det \left(\begin{array}{c|c} a_{11} & * \\ \hline 0 & \\ \vdots & A_{11} \\ 0 & \end{array} \right) = a_{11} \det A_{11}$

(21.2) • $\det \begin{pmatrix} a_{11} & a_{22} & & * \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}$

(21.3) • $\det A^T = \det A$ $\quad \forall A \in M_{n,n}$

(21.4) • $\det(AB) = \det A \cdot \det B$ $\quad \forall A, B \in M_{n,n}$.

Lemma 22.1 An $n \times n$ matrix A is invertible $\iff \det A \neq 0$.

Proof ($\Rightarrow$) Suppose A is invertible. Then $\exists A^{-1}$ s.t.
$AA^{-1} = I$

$\Rightarrow 1 = \det I = \det A \cdot \det A^{-1}$ $\quad \Rightarrow \det A \neq 0$.

($\Leftarrow$) Suppose $A \in M_{n,n}$ is ~~not~~ not invertible. Then $\operatorname{rank} A < n$
and $\operatorname{null}(A) > 0$. $\Rightarrow$ columns of A are linearly dependent.

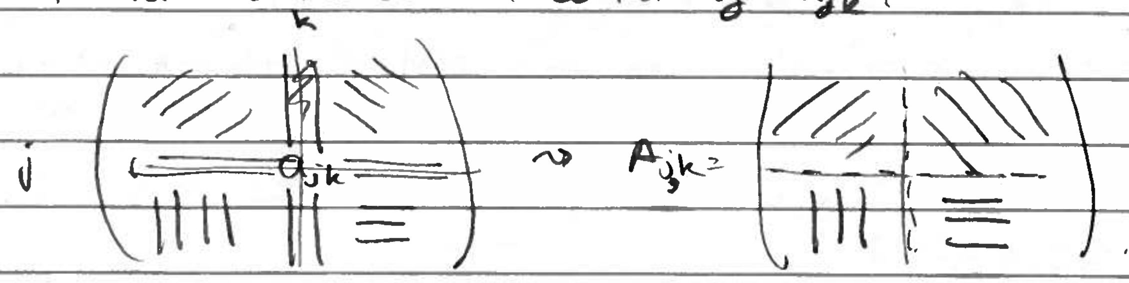
$\Rightarrow \det A = 0$ by a homework problem:

($\exists i \in \{1, \ldots, n\}$ so that A's i^{th} column a_i is a linear
combination of the rest of the columns: $a_i = \sum_{j \neq i} c_j a_j$

$\Rightarrow \det A = \det(a_1, \ldots a_{i-1}, \sum c_j a_j, a_{i+1}, \ldots a_n) = 0$.

Notation Let A be an $n\times n$ matrix

$A_{j,k}$ = $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and the column containing a_{jk}.



Note clash of notation: here $A_{j,k}$ is not jk^{th} entry of A!

$$\text{Ex}\quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{1,2} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$$

Lemma 22.2 For any $n\times n$ matrix $A=(a_{ij})$

$$\det A = \sum_{k=1}^{n} (-1)^{k+1} a_{k1} \det(A_{k,1})$$

Proof

$$\det\begin{pmatrix} a_{11} & \\ a_{21} & \\ \vdots & \\ a_{n1} & \end{pmatrix} = \det\begin{pmatrix} a_{11} & \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} + \det\begin{pmatrix} 0 & \\ a_{21} & \\ 0 & \\ \vdots & \end{pmatrix} + \cdots + \det\begin{pmatrix} 0 & \\ \vdots & \\ 0 & \\ a_{n1} & \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{11} & \\ 0 & A_{11} \\ \vdots & \end{pmatrix} - \det\begin{pmatrix} a_{21} & \\ 0 & A_{2,1} \\ \vdots & \end{pmatrix} + \det\begin{pmatrix} a_{31} & \\ 0 & A_{3,1} \\ \vdots & \end{pmatrix} + \cdots$$

$$+ (-1)^{n+1}\det\begin{pmatrix} a_{n1} & \\ 0 & A_{n,1} \\ \vdots & \\ 0 & \end{pmatrix} = \sum_{k=1}^{n} (-1)^{k+1} a_{k1} \det(A_{k,1})$$

↑ ~~Here we use 21.1~~ ☐

Remark

Similarly one can show that

$$(**)\quad \det A = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} \det(A_{1,k})$$

which is what the textbook uses as a definition of det.

More generally, $\boxed{(+)\quad \det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{i,j})}$

Definition (cofactor) Let $A=(a_{ij})$ be an $n\times n$ matrix.

The matrix C of cofactors is the $n\times n$ matrix with entries

$$C_{ij} = (-1)^{i+j} \det(A_{i,j})$$

<u>Theorem 22.3</u> (Treil, thm 5.2 on p92)
Let A be an invertible $n \times n$ matrix, and C its cofactor matrix.

$$A^{-1} = \frac{1}{\det A} C^T.$$

<u>Proof</u> Recall that for any two $n \times n$ matrices $B = (b_{ik})$, $D = (d_{kj})$

$$(BD)_{ij} = \sum_k b_{ik} d_{kj}$$

Now

$$(AC^T)_{ij} = \sum_k a_{ik} (C^T)_{kj} = \sum_k a_{ik} C_{jk}$$

If $i=j$ we get $(AC^T)_{ii} = \sum_k a_{ik} C_{ik} = \sum a_{ik} (-1)^{i+k} \det(A_{i,k})$
$= \det A.$ by (*).

If $i \neq j$ we set

$$(AC^T)_{ij} = \sum a_{ik} (-1)^{j+k} \det(A_{j,k})$$

$$= \det \begin{pmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{pmatrix} \leftarrow j^{th} \text{ row !}$$

$\Rightarrow$ So $(AC^T)_{ij} = 0$ since i^{th} and j^{th} row are the same.
We conclude that

$$AC^T = \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix} = \det A \cdot I.$$

$$\Rightarrow A \cdot \left(\frac{1}{\det A} C^T \right) = I \Rightarrow A^{-1} = \frac{1}{\det A} C^T.$$

<u>Ex</u> $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad C = \begin{pmatrix} (-1)^{1+1} a_{22} & (-1)^{1+2} a_{21} \\ (-1)^{1+2} a_{12} & (-1)^{2+2} a_{11} \end{pmatrix}$

$$\Rightarrow C^T = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Definition Two $n \times n$ matrices A & B are <u>similar</u> if $\exists$ an invertible matrix S so that

$$A = SBS^{-1}.$$

"Example" Suppose $T: V \to V$ is linear, $\mathcal{A}, \mathcal{B}$ two bases of V. Then $[T]_{\mathcal{A}\mathcal{A}}$ and $[T]_{\mathcal{B}\mathcal{B}}$ are similar.

Reason

$$[T]_{\mathcal{A}\mathcal{A}} [id]_{\mathcal{A}\mathcal{B}} = [T]_{\mathcal{A}\mathcal{B}} = [id]_{\mathcal{A}\mathcal{B}} [T]_{\mathcal{B}\mathcal{B}}$$

$$\Rightarrow [T]_{\mathcal{A}\mathcal{A}} = [id]_{\mathcal{A}\mathcal{B}} [T]_{\mathcal{B}\mathcal{B}} \left([id]_{\mathcal{A}\mathcal{B}}\right)^{-1}.$$

Lemma 22.4 Suppose A & B are two similar matrices. Then $\det A = \det B$.

Proof Since A & B are similar, $A = SBS^{-1}$ for some S.

$$\Rightarrow \det A = \det(SBS^{-1}) = \det(S) \det B \det(S^{-1})$$
$$= \det(S) \det(S^{-1}) \det B$$
$$= \det(I) \det B = \det B.$$

Lemma 22.5 Let $T: V \to V$ be a linear map. Then $\det([T]_{\mathcal{A}\mathcal{A}})$ does not depend on the choice of a basis $\mathcal{A}$:

$$\det([T]_{\mathcal{A}\mathcal{A}}) = \det([T]_{\mathcal{B}\mathcal{B}})$$

for any two bases $\mathcal{A}$ and $\mathcal{B}$.

Proof "Example" and 22.4.