Last time

1) constructed a map $\rho : S_n \to M_{n,n}(\mathbb{R})$

so that $\rho(\sigma\mu) = \rho(\sigma)\rho(\mu)$

$$\rho((ij)) = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \begin{matrix} i \\ j \end{matrix} \quad (i<j).$$

2) Constructed $\text{sign}: S_n \to \{\pm 1\}$ so that

$\text{sign}(\sigma\mu) = \text{sign}(\sigma)\,\text{sign}(\mu) \quad \forall \sigma, \mu$

$\text{sign}((ij)) = -1.$

3) Defined $\det_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} = M_{n,n}(\mathbb{R}) \to \mathbb{R}$ by

$$(*) \quad \det((a_{ij})) = \sum_{\sigma \in S_n} (\text{sign}\,\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

and observed (i) det is n-linear

(ii) $\forall D : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R}$ n-linear and alternating

$D((a_{ij})) = \det(a_{ij}) \cdot D(e_1 \ldots e_n)$

Theorem 20.5 $(*)$ defines an alternating map.

Proof Fix a matrix (a_{ij}). Let (b_{ij}) be the matrix obtained from (a_{ij}) by switching rows columns k and ℓ

Then $\forall i,j$

$$b_{ij} = a_{ij} \quad \text{if } i \neq k, \ell$$

$$b_{kj} = a_{\ell j}$$

$$b_{\ell j} = a_{kj}.$$

$\Rightarrow \forall (i,j) \quad b_{ij} = a_{\tau(i)j} \quad \text{where } \tau = (k, \ell).$

$$\det((b_{ij})) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, b_{\sigma(1)1} \cdots b_{\sigma(n)n}$$

$$= \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{\tau\sigma(1)1}\, a_{\tau\sigma(2)2} \cdots a_{\tau\sigma(n)n}$$

("Change of variables" $\mu := \tau\sigma$. Then $\sigma = \tau^{-1}\mu$

$$\Rightarrow \det((b_{ij})) = \sum_{\mu \in S_n} \text{sign}(\tau^{-1}\mu)\, a_{\mu(1)1} \cdots a_{\mu(n)n}$$

Note $\quad \operatorname{sign}(\tau^{-1}\mu) = \operatorname{sign}(\tau^{-1}) \operatorname{sign}\mu = -\operatorname{sign}\mu.$

$\Rightarrow \det((b_{ij})) = -\sum_{\mu \in S_n} \operatorname{sign}(\mu)\, a_{\mu(1)1} \cdots a_{\mu(n)n}.$

$\boxed{\text{Q}_1 \ \text{Why is } \det(e_1, \dots, e_n) = 1 \ ?}$

Note: Since det is alternating, $\forall a_1 \dots a_n \in \mathbb{R}^n$, $\forall i, j$, $\forall \lambda \in \mathbb{R}$

$$\det(a_1, \dots, a_i + \lambda a_j, a_{i+1} \dots a_j \dots a_n)$$

$$= \det(a_1 \dots a_i, a_{i+1} \dots a_n) + \lambda \det(a_1 \dots \underset{i^{\text{th}} \text{ slot}}{a_j} \dots a_j \dots a_n)$$

$$= \det(a_1 \dots a_n) + \lambda \cdot 0$$

"adding a multiple of j^{th} column to i^{th} column does not change determinant".

Lemma 21.1 Let A be an $n \times n$ matrix of the form

$$A = \begin{pmatrix} a_{11} & * \\ 0 & \\ \vdots & A_{11} \\ 0 & \end{pmatrix}, \quad \text{where } A_{11} \in M_{n-1,n-1}(\mathbb{R}).$$

Then $\det_n(A) = a_{11} \det_{n-1}(A_{11})$.

Proof ① $\det A = a_{11} \det \begin{pmatrix} 1 & * \\ 0 & \\ \vdots & A_{11} \\ 0 & \end{pmatrix}$ by n-linearity.

② $\det \begin{pmatrix} 1 & * \\ 0 & \\ \vdots & A_{11} \\ 0 & \end{pmatrix} = \det \begin{pmatrix} 1 & 0\,0\,0 \cdots 0 \\ 0 & \\ \vdots & A_{11} \\ 0 & \end{pmatrix}$ by Note above.

③ $D : \mathbb{R}^{n-1} \times \dots \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$

$D(B) = \det_n \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$: is $n-1$ linear and alternating.

$$\Rightarrow D(B) = \det_{n-1}(B)\, D\underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{n-1} = \det_{n-1}(B) \det_n \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$= \det_{n-1}(B) \cdot 1 .$$

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Corollary 21.2 If an $n \times n$ matrix A is of the form

$$A = \begin{pmatrix} a_{11} & & * \\ & a_{22} & \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$ then $\det A = a_{11} a_{22} \cdots a_{nn}$.

Proof Lemma 21.1 + induction on n.

Lemma 21.3 For any $n \times n$ matrix A
$$\det(A^T) = \det A.$$

Proof (i) Note first that $\forall \sigma \in S_n$
$$1 = \text{sign}(id) = \text{sign}(\sigma\sigma^{-1}) = \text{sign}(\sigma) \cdot \text{sign}(\sigma^{-1}).$$
$\Rightarrow$ Since $\text{sign}(\sigma) = \pm 1$, $\boxed{\text{sign}\,\sigma^{-1} = \text{sign}\,\sigma}$

(ii) The map $inv : S_n \to S_n$ $inv(\sigma) = \sigma^{-1}$ is a bijection.

(iii)
$$\det(A^T) = \sum_{\sigma \in S_n} \text{sign}(\sigma) (A^T)_{\sigma(1)1} \cdots (A^T)_{\sigma(n)n}$$
$$= \sum_{\sigma \in S_n} \text{sign}(\sigma) A_{1\sigma(1)} \cdots A_{n\sigma(n)}$$

("Change of variables" If $j = \sigma(i)$ then $A_{i\sigma(i)} = A_{\sigma^{-1}(j)j}$)
$$\Rightarrow \prod_{i=1}^{n} A_{i\sigma(i)} = \prod_{j=1}^{n} A_{\sigma^{-1}(j)j}$$
$$\Rightarrow \det(A^T) = \sum_{\sigma \in S_n} \text{sign}(\sigma^{-1}) \prod_{j=1}^{n} A_{\sigma^{-1}(j)j} = (\mu = \sigma^{-1})$$
$$= \sum_{\mu \in S_n} \text{sign}(\mu) \cdot \prod_{j=1}^{n} A_{\mu(j)j} = \det A. \quad \square$$

Consequences

$\det A$ is an alternating n-linear function of rows of A (which are columns of A^T).

So

(1) interchanging two rows changes sign of det

(2) multiplying a row by a scalar multiplies the determinant by a scalar

(3) adding a multiple of one row to another row doesn't change the determinant.

Ex $\forall x, y \in \mathbb{R}$ $\det\begin{pmatrix} 1 & x \\ 1 & y \end{pmatrix} = \det\begin{pmatrix} 1 & x \\ 0 & y-x \end{pmatrix} = 1 \cdot (y-x)$.

$$\det\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix} = -\det\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix} = (-1)^2 \det\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & -6 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{pmatrix} = -12$$

Fact Suppose $\alpha : \overbrace{W \times \cdots \times W}^{k} \to \mathbb{R}$ is k-linear and alternating and $T : V \to W$ is a linear map. Then $T^*\alpha : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$ defined by

$$(T^*\alpha)(v_1 \ldots v_k) = \alpha(Tv_1, \ldots Tv_k)$$

is k-linear and alternating.

Proof Homework.

Theorem 21.4 For any $A, B \in M_{n,n}(\mathbb{R})$

$$\det(AB) = \det A \det B.$$

Proof (i) $A^*\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ $(A^*\det)(v_1 \ldots v_n) = \det(Av_1 | \ldots | Av_n)$ is n-linear and alternating.

(ii) $A^*\det(\cdot) = (A^*\det)(e_1, \ldots e_n) \cdot \det(\cdot)$

Since $\forall D : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ n-lin and alternating

$D = D(e_1 \ldots e_n) \cdot \det$.

$$\underbrace{(A^*\det)(B)}_{\det(Ab_1 \ldots Ab_n)} = \underbrace{(A^*\det)(e_1 \ldots e_n)}_{\det(Ae_1 | \ldots Ae_n)} \cdot \det B$$

$\therefore$ $\det(AB) = \det A \det B$.