03/8/2019
Fr

20.1

Goals for today:

(1) Prove that there is a map $\mathrm{sign}: S_n \to \{\pm 1\}$ so that if $\sigma = \tau_1 \dots \tau_k$ for some transpositions $\tau_1 \dots \tau_k$ then
$$\mathrm{sign}(\sigma) = (-1)^k$$

$\Rightarrow$

(2) Prove that $\det: \mathbb{R}^n \times \dots \times \mathbb{R}^n = M_{n,n}(\mathbb{R}) \to \mathbb{R}$
$$\det(a_{ij}) := \sum_{\sigma \in S_n} (\mathrm{sign}\,\sigma)\, a_{\sigma(1)1} \dots a_{\sigma(n)n}$$
is an alternating n-linear map

Consequence of (1) and (2):

Recall: We proved last time $\forall \sigma \in S_n$ $\exists$ transpositions $\tau_1 \dots \tau_k$ s.t. $\sigma = \tau_k \dots \tau_1$ and

for any alternating n-linear map $D: M_{n,n}(\mathbb{R}) \to \mathbb{R}$
$$D((a_{ij})) = \Big(\sum_{\substack{\sigma \in S_n \\ \sigma = \tau_k \dots \tau_1}} (-1)^k a_{\sigma(1)1} \dots a_{\sigma(n)n}\Big) D(e_1, \dots, e_n)$$
$$= \underbrace{\Big(\sum_{\sigma \in S_n} \mathrm{sign}\,\sigma\; a_{\sigma(1)1} \dots a_{\sigma(n)n}\Big)}_{\det(a_{ij})} D(e_1 \dots e_n)$$

~~Hence (1)~~ $\forall D \in \mathrm{Alt}^n(\mathbb{R}^n)$ $\quad D = D(e_1 \dots e_n) \det$

$\Rightarrow \mathrm{Alt}^n(\mathbb{R}^n) = \mathrm{span}\{\det\}$.

Lemma 20.1 There is a map $\rho: S_n \to M_{n,n}(\mathbb{R})$ with $\rho(\sigma\mu) = \rho(\sigma)\rho(\mu)$ $\forall \sigma, \mu \in S_n$ and
$$\rho((ij)) = \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \end{array}$$

Proof A linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is uniquely determined by what it does to the standard basis.

Define $\rho(\sigma)$, by $\rho(\sigma)(e_i) := e_{\sigma(i)}$ $\forall i$

Then $\rho((ij)) = \begin{pmatrix} 1 & & & \\ & 0 & & 1 \\ & & 1 & \\ & 1 & & 0 \\ & & & & 1 \end{pmatrix}$.

Note: $\forall\, \sigma, \mu \in S_n \quad (\sigma \circ \mu)(i) = \sigma(\mu(i))$.

$$\Rightarrow \rho(\sigma)\,(\rho(\mu)\,(e_i)) = \rho(\sigma)\, e_{\mu(i)} = e_{\sigma(\mu(i))} = e_{(\sigma\circ\mu)(i)}$$
$$= \rho(\sigma\circ\mu)\,(e_i).$$

ρ is called the permutation representation: it "represents" permutations as matrices.

Note: $\forall\, x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $\rho(\sigma)(x) = \rho(\sigma)\left(\sum x_i e_i\right)$

$$= \sum_{i=1}^n x_i e_{\sigma(i)} \quad \left(\begin{array}{l}\text{"change of variables"} \\ j = \sigma^{-1}(i) \\ i = \sigma^{-1}(j)\end{array}\right) = \sum_{i=1}^n x_{\sigma^{-1}(i)} e_i$$

So $\rho(\sigma)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}$

Now define a polynomial $\Delta_n : \mathbb{R}^n \to \mathbb{R}$ by

$$\Delta_n(x_1 \ldots x_n) = \prod_{i<j} (x_i - x_j)$$

Ex
$n=2 \quad \Delta(x_1, x_2) = x_1 - x_2$
$n=3 \quad \Delta(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$
$n=4 \quad \Delta(x_1, x_2, x_3, x_4) = (x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)$.

Definition We define sign: $S_n \to \{\pm 1\}$ by

$$\text{sign}(\sigma) = \frac{\Delta(\rho(\sigma)(x_1 \ldots x_n))}{\Delta(x_1 \ldots x_n)} = \frac{\prod_{i<j}\left(x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(j)}\right)}{\prod_{i<j}(x_i - x_j)}$$

Ex $\quad n=3 \quad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \left(\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\right)$

$$\rho(\sigma)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ x_{\sigma^{-1}(3)} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}$$

$$\text{sign}\,\sigma = \frac{\Delta(x_3, x_1, x_2)}{\Delta(x_1, x_2, x_3)} = \frac{(x_3 - x_1)(x_3 - x_2)(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} = (+1)^2 = 1$$

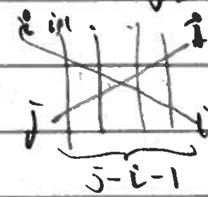
Definition Let σ be a permutation. An inversion is a pair i, j s.t. $i < j$ and $\sigma(i) > \sigma(j)$.

The inversion number of σ = # of inversions in σ.

Ex $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ inversion no. of $\sigma = 2$

Note (i) inv. no. σ = inv. no. of σ^{-1}, and $\text{sign}(\sigma) = (-1)^{\text{inv. no. } \sigma^{-1}}$

(ii) if τ is a transposition (ij)

then inv. no. of τ =  $= 2(j-i-1)+1$

$\Rightarrow \text{sign}(ij) = -1.$

Lemma 20.2 $\forall\, \sigma, \mu \in S_n$ $\text{sing}(\sigma\mu) = \text{sign}\,\sigma \cdot \text{sing}\,\mu$

Proof

$$\text{sign}(\sigma\mu) = \frac{\Delta(\rho(\sigma\mu)x)}{\Delta(x)} = \frac{\Delta(\rho(\sigma)\underbrace{\rho(\mu)x}_{y})}{\Delta(\underbrace{\rho(\mu)x}_{y})} \cdot \frac{\Delta(\rho(\mu)x)}{\Delta(x)}$$

$$= \frac{\Delta(\rho(\sigma)y)}{\Delta(y)} \cdot \frac{\Delta(\rho(\mu)x)}{\Delta(x)} = \text{sign}(\sigma)\,\text{sign}(\rho).$$ ∎

Corollary 20.3 Suppose $\sigma \in S_n$, $\tau_1 \ldots \tau_k$ transpositions and $\sigma = \tau_1 \ldots \tau_k$. Then $\text{sign}(\sigma) = (-1)^k$.

Proof $\text{sign}(\sigma) = \text{sign}(\tau_1) \ldots \text{sign}(\tau_k) = (-1)^k$.

Lemma 20.4 For any $\mu \in S_n$ the map $L_\mu : S_n \to S_n$ $L_\mu(\sigma) = \mu\sigma$ is a bijection with the inverse given by $L_{\mu^{-1}}$.

Proof $\forall\, \sigma \in S_n$

$$L_{\mu^{-1}}(L_\mu(\sigma)) = \mu^{-1}(\mu\sigma) = \text{id}_{\{1,\ldots,n\}} \circ \sigma = \sigma$$

$$L_\mu(L_{\mu^{-1}}(\sigma)) = \mu\mu^{-1}\sigma = \text{id}_{\{1,\ldots,n\}}\,\sigma = \sigma$$

$$\therefore\ L_{\mu^{-1}} \circ L_\mu = \text{id}_{S_n} = L_\mu \circ L_{\mu^{-1}}.$$

Theorem 20.5 The map $\det : \mathbb{R}^{n^2} = \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n} = M_{n,n}(\mathbb{R}) \to \mathbb{R}$

$$\det((a_{ij})) = \sum_{\sigma \in S_n} \text{sign}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

is an alternating n-linear map with $\det\left(\begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix}\right) = 1$.

Proof $\forall \sigma \in S_n$ the map $\mathbb{R}^{n^2} \longrightarrow \mathbb{R}$

$$(a_{ij}) \longmapsto \operatorname{sign}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

is n-linear.

$$\Rightarrow \det((a_{ij})) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

is n-linear.

We now argue that det is alternating. Fix a matrix (a_{ij}). Let (b_{ij}) be the matrix obtained from (a_{ij}) by switching k^{th} and l^{th} columns. Then $\forall i,j$

$$b_{ij} = a_{ij} \text{ if } i \neq k, l$$
$$b_{kj} = a_{lj}$$
$$b_{lj} = a_{kj}$$

$\forall i,j \Rightarrow b_{ij} = a_{\tau(i)j}$ where $\tau = (k, l)$

$$\det((b_{ij})) = \sum_{\sigma \in S_n} \operatorname{sign}\sigma\ b_{\sigma(1)1} \cdots b_{\sigma(n)n}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{\tau(\sigma(1))1}\, a_{\tau(\sigma(2))2} \cdots a_{\tau(\sigma(n)), n}$$

(since $L_\tau : S_n \to S_n$, $\sigma \mapsto \tau\sigma$ is a bijection, we can change variables:

$\tau\sigma = \mu$ $\quad \sigma = \tau^{-1}\mu = \tau\mu$ $\quad (\tau = \tau^{-1}!)$)

$$\Rightarrow \det((b_{ij})) = \sum_{\mu \in S_n} \operatorname{sign}(\tau\mu)\, a_{\mu(1)1} a_{\mu(2)2} \cdots a_{\mu(n)n}$$

$$= (\operatorname{sign}\tau) \sum_{\mu \in S_n} \operatorname{sign}(\mu)\, a_{\mu(1)1} \cdots a_{\mu(n)n}$$

$$= (-1) \det((a_{ij}))$$

$\therefore$ det is alternating. $\square$

Next time $\forall A, B \in M_{n \times n}(\mathbb{R})$ $\quad \det(AB) = \det A \det B$