Last time

- Solving a system of linear equations by Gaussian elimination
- $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$ is equivalent to $(a_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Even more succinctly: (**) $A\vec{x} = \vec{b}$

- In terms of maps, (**) says $\vec{x}$ solves (**) $\Leftrightarrow T_A(\vec{x}) = \vec{b}$ where $T_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum a_{1i}x_i \\ \vdots \\ \sum a_{mi}x_i \end{pmatrix}$
- We checked that $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is linear: $\forall \lambda, \mu \in \mathbb{R}$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$

$$T_A(\lambda\vec{x} + \mu\vec{y}) = \lambda T_A(\vec{x}) + \mu T_A(\vec{y})$$

Note $f : \mathbb{R} \to \mathbb{R}$ $f(x) = ax + b$, $a, b \in \mathbb{R}$, $a, b \neq 0$, is **not** linear. Why not?

Today: Vector spaces (over $\mathbb{R}$).

"Recall" We have two operations on elements of $\mathbb{R}^n = \{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}\}$:

1) multiplication by scalars

$$\lambda \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} \quad \forall \lambda \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^n$$

2) addition

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

These two operations (which are just functions $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$) has several properties. We single out 8:

1) $+$ is commutative: $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$
2) $+$ is associative: $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ for all $\vec{x},\vec{y},\vec{z} \in \mathbb{R}^n$.
3) There is a vector $\vec{0}$ (zero vector) so that
$$\vec{x}+\vec{0} = \vec{x} = \vec{0}+\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n$$
4) additive inverses: $\forall \vec{x} \in \mathbb{R}^n \ \exists -\vec{x} \in \mathbb{R}^n$ s.t. $\vec{x}+(-\vec{x}) = \vec{0}$.
5) multiplication by 1 doesn't do anything: $1\cdot\vec{x} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$
6) $\lambda\cdot(\mu\cdot\vec{x}) = (\lambda\mu)\cdot\vec{x} \quad \forall \lambda,\mu \in \mathbb{R},\ \vec{x} \in \mathbb{R}^n$
7) $\lambda\cdot(\vec{x}+\vec{y}) = \lambda\cdot\vec{x}+\lambda\cdot\vec{y} \quad \forall \lambda \in \mathbb{R},\ \forall \vec{x},\vec{y} \in \mathbb{R}^n$
8) $(\lambda+\mu)\cdot\vec{x} = (\lambda\cdot\vec{x})+(\mu\cdot\vec{x}) \quad \forall \lambda,\mu \in \mathbb{R},\ \vec{x} \in \mathbb{R}^n$.

<u>Definition</u> A (real) <u>vector space</u> (or a vector space over the field of real numbers) is a set V together with two functions
$$\mathbb{R}\times V \to V, \quad (\lambda,\vec{x}) \mapsto \lambda\cdot\vec{x} \qquad +: V\times V \to V, \quad (\vec{x},\vec{y}) \mapsto \vec{x}+\vec{y}$$
and an element $\vec{0} \in V$ so that properties (1)–(8) above hold for all $\lambda,\mu \in \mathbb{R}$ and all $\vec{x},\vec{y},\vec{z} \in V$.

<u>Ex</u> $\mathbb{R}^n$ $(n=0,1,2,\ldots)$ are vector spaces

<u>Note</u> $\mathbb{R}^0 = \{0\}$ $\quad +: \{0\}\times\{0\} \to \{0\}$ is the only map
and $\mathbb{R}\times\{0\} \to \{0\}$ is the only map

<u>Ex</u> $\mathbb{R}[x] = \{a_0 + a_1 x + \ldots + a_n x^n \mid n \geq 0,\ a_0,\ldots,a_n \in \mathbb{R}\}$
= polynomials with real coefficients
is a vector space over $\mathbb{R}$. under the usual addition of polynomials and multiplication by scalars

<u>Ex</u> $C^0([0,1]) = \{f: [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$
is a vector space under the usual addition of functions and multiplication by scalars.

Nonexample $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x, y \geq 0 \right\}$

is not a vector space: while $V \times V \to V$

$+ : \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \longmapsto \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}$ makes sense

$\mathbb{R} \times V \to V \qquad \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$ does not:

$(-1) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \notin V.$

Lemma 2.1 In a vector space V, the zero vector is unique: if $0' + \vec{v} = \vec{v}$ for all $\vec{v} \in V$, then $0' = \vec{0}$.

Proof $0' + \vec{0} = \vec{0}$ by definition of $0'$

$0' + \vec{0} = 0'$ by definition of $\vec{0}$.

$\Rightarrow \vec{0}' = 0' + \vec{0} = \vec{0}.$ $\square$

Lemma 2.2 Let V be a vector space. For any $\vec{v} \in V$

$$0 \cdot \vec{v} = \vec{0}.$$

Proof $0 \cdot \vec{v} = (0 + 0) \cdot \vec{v} = 0 \cdot \vec{v} + 0 \cdot \vec{v}.$

Now add $-(0 \cdot \vec{v})$ to both sides ($-(0 \cdot \vec{v})$ exists by (4))

We get $0 \cdot \vec{v} + (-(0 \cdot \vec{v})) = 0 \cdot \vec{v} + (0 \cdot \vec{v} + (-(0 \cdot \vec{v})))$

$\Rightarrow \quad \vec{0} = 0 \cdot \vec{v} + \vec{0} = 0 \cdot \vec{v}$ $\square$

Lemma 2.3 Let V be a vector space.

1) Additive inverses are unique: if $\vec{u} + \vec{v} = \vec{0}$, then $\vec{u} = -\vec{v}$.

2) $(-1) \cdot \vec{v} = -\vec{v}$.

Proof (1) $-\vec{v} \overset{(3)}{=} \vec{0} + (-\vec{v}) \overset{\text{assumption}}{=} (\vec{u} + \vec{v}) + (-\vec{v}) \overset{\text{associativity}}{=} \vec{u} + (\vec{v} + (-\vec{v})) \overset{(4)}{=} \vec{u} + \vec{0} \overset{(3)}{=} \vec{u}$

2) $\vec{v} + (-1) \cdot \vec{v} = 1 \cdot \vec{v} + (-1) \cdot \vec{v} = (1 + (-1)) \vec{v} = 0 \cdot \vec{v} = \vec{0}$ (by 2.2) $\square$

Definition (subspace) A subset W of a vector space V is a subspace if W a vector space under the operations of $+$ and scalar multiplication that are defined on V.

Ex $V = \mathbb{R}^2$ $W = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 = x_2\}$ is a subspace:
$= \{\begin{pmatrix} x \\ x \end{pmatrix} \mid x \in \mathbb{R}\}$.

Check: (i) $\forall \lambda \in \mathbb{R}$ $\lambda \cdot \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda x \end{pmatrix} \in W$

So we have $\mathbb{R} \times W \longrightarrow W$

(ii) $\forall \begin{pmatrix} x \\ x \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix} \in W$ $\begin{pmatrix} x \\ x \end{pmatrix} + \begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix} \in W$

So we have $+ : W \times W \to W$.

(iii) $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W$

(iv) $\forall \begin{pmatrix} x \\ x \end{pmatrix} \in W$, $-\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} -x \\ -x \end{pmatrix} \in W$

Note that $+$ is automatically commutative and associative

Also $\forall \vec{x} \in W$ $1 \cdot \vec{x} = \vec{x}$ since it's true $\forall \vec{x} \in \mathbb{R}^2$

Similarly (6), (7), (8) hold automatically.

Non-example $W = \mathbb{R}^2 \setminus \{\vec{0}\}$ is not a subspace of $\mathbb{R}^2$: $\vec{0} \notin W$.

Non-example $W = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 1\}$ is not a subspace of $\mathbb{R}^2$: it's not closed under $+$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin W$.

Ex For each $n \geq 0$, $P_n = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, \ldots, a_n \in \mathbb{R}\}$
$= \{p(x) \in \mathbb{R}[x] \mid \deg p \leq n\}$ is a subspace of $\mathbb{R}[x]$:
P_n is closed under $+$ and scalar multiplication,
the zero polynomial is in P_n and if $p(x) \in P_n$, $-p(x) \in P_n$ as well.

Definition A map $T : V \to W$ between two vector spaces is linear if T preserves $+$ and scalar multiplication:
$\forall \lambda, \mu \in \mathbb{R}$, $\vec{x}, \vec{y} \in V$.

$$T(\lambda \vec{x} + \mu \vec{y}) = \lambda T(\vec{x}) + \mu T(\vec{y})$$

Example We have seen that any $m\times n$ matrix $A=(a_{ij})$ defines a linear map $T_A: \mathbb{R}^n \to \mathbb{R}^m$

$$T_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \sum_i a_{1i} x_i \\ \vdots \\ \sum_i a_{mi} x_i \end{pmatrix}$$

Ex $\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$,

$\frac{d}{dx}\left(\sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} i a_i x^{i-1}$ is linear.

Question Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map.
Is there an $m\times n$ matrix $A=(a_{ij})$ st

$T(\vec{x}) = T_A(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$?